

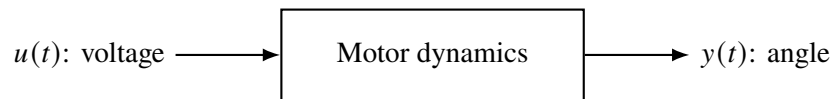
LECTURES 01–04 AND THE DRONE ACTIVITY

Signals, state and feedback

This 20-page practice companion keeps the main explanations and worked examples. Read one topic, cover the solution, and explain the steps yourself. For the separate 40-question practice set, use pp. 1–2 for system basics, pp. 3–5 for Laplace, pp. 7–10 and 20 for modeling, and pp. 12–14 for block diagrams. Additional written practice and answers are at the end.

Start by defining the input and output

A dynamic system changes over time. For a motor, voltage is a possible input, and shaft angle is a possible output:



The command alone does not determine the motion. At zero voltage, the shaft might be stopped or still spinning. The **state** records the present condition needed, with the model and future inputs, to predict what happens next. For this mechanical example, use angle and angular velocity.

Three distinctions to keep separate

Question	Meaning
SISO or MIMO?	Count the chosen input and output signals, not the number of machines. One voltage-to-angle channel is SISO.
Time-invariant?	The governing rule has no explicit dependence on absolute time. A pendulum can move while its rule stays unchanged.
Continuous or discrete?	$\theta(t)$ describes physical motion; $\theta[k] = \theta(kT_s)$ is a sampled record. T_s is the time between samples; sample k occurs at $t = kT_s$. Sampling the record does not make the physical motion discrete.

Control makes the input depend on what we want

The reference $r(t)$ is the desired output; $y(t)$ is the actual output. Their difference is the error:

$$e(t) = r(t) - y(t).$$

A feedback controller uses the measured error to choose the next input. For example, if a motor has not reached its desired angle, the controller can apply a correcting voltage. Desired angle and measured angle need not match during motion.

Keep these roles distinct

Input: what we apply. Output: what we observe. State: what the system remembers about its present motion.
Reference: what we want the output to do.

[Course pages and interactive examples](#). This review stops at the drone activity; the later first-order-response lesson is not included. It does not change announced exam coverage or permitted materials.

Linearity and useful local approximations

Linear means that scaling and addition both work

For a linear input–output map, multiplying an input by a number multiplies the response by the same number. Adding two inputs adds their responses:

$$u_1 \mapsto y_1, \quad u_2 \mapsto y_2 \quad \implies \quad \alpha u_1 + \beta u_2 \mapsto \alpha y_1 + \beta y_2.$$

This is **superposition**. For a dynamic input–output test, use zero initial conditions; a nonzero starting-state response must be treated separately.

A quick test

For $y = 3u$, doubling u doubles y , and sums also work: this is linear. For $y = u^2$, doubling the input gives

$$(2u)^2 = 4u^2, \quad \text{not } 2u^2.$$

So $y = u^2$ is nonlinear. Even $y = 3u + 1$ is not linear: zero input produces output 1. A straight-looking graph alone is not a superposition test.

Linearity and time dependence are different questions. With fixed g and ℓ , $\ddot{\theta} = -(g/\ell) \sin \theta$ is nonlinear but time-invariant. A model $\dot{x} = -a(t)x + u$ with a prescribed changing coefficient is linear but time-varying.

Why a nonlinear spring can look linear nearby

A car's weight already compresses its suspension. Call that compression x_0 . A small bump adds a small amount Δx . Over a short part of the force curve, its slope is nearly constant:

$$\underbrace{F(x_0 + \Delta x) - F(x_0)}_{\text{additional force } \Delta F} \approx \underbrace{F'(x_0)}_{\text{local stiffness } k_{\text{local}}} \Delta x.$$

If the local stiffness is 10 N/cm, an additional compression of 0.2 cm gives about 2 N of *additional* force. It does not give the total force supporting the car.

For a deep compression, the slope may change appreciably. Then one fixed stiffness no longer predicts the force well over the entire motion. The spring has not become globally linear; we have built a useful local model around its normal operating point.

The idea we will reuse

Choose an operating point. Describe small changes from that point. Keep the local slope, and check that the motion stays in a range where the approximation is useful.

Laplace Transform: From Time to Algebra

A robot can keep moving after a push ends. Its differential equation describes that motion, including the effect of the starting state. For the linear differential equations we study, the Laplace transform turns the calculation into algebra. We solve in s , then transform back to find the motion. It changes how we calculate, not the physical system; it does not make a nonlinear model linear.

For a zero-initial-state LTI system, the transformed input-output relation is $Y(s) = G(s)U(s)$: input, system, and response remain distinct. Nonzero initial conditions contribute additional terms, as shown later.

The definition is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt.$$

Here $s = \sigma + j\omega$, with $j^2 = -1$, is a complex variable, not another name for time. Since $e^{-st} = e^{-\sigma t}e^{-j\omega t}$, its real part σ controls exponential weighting and its imaginary part ω describes oscillation. Engineers usually write j rather than i for the imaginary unit.

Worked example: the unit step

The unit step $u_s(t)$ equals zero before $t = 0$ and one from $t = 0$ onward. Because the integral starts at zero, replace the step by one:

$$\begin{aligned}\mathcal{L}\{u_s(t)\} &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_0^T = \lim_{T \rightarrow \infty} \frac{1 - e^{-sT}}{s} \\ &= \frac{1}{s}, \quad \text{Re}(s) > 0.\end{aligned}$$

The condition matters: $|e^{-sT}| = e^{-\sigma T}$ tends to zero for $\sigma > 0$. We did not remove the step arbitrarily; it equals one throughout the integration interval. Writing an upper limit T first makes the limit at infinity explicit.

Return to a physical command

A step of size A has transform A/s . For a motor-speed command $r(t) = 90u_s(t)$, $R(s) = 90/s$. The command jumps and stays at 90; that does not mean the motor's measured speed jumps. The system dynamics determine how the output responds.

The next page collects useful transform pairs, so we do not need to repeat this integral for every signal. A transform is not obtained by simply replacing t with s .

Use the Table in Both Directions

These signals are zero before $t = 0$. The impulse $\delta(t)$ represents a very short action with unit area, not a finite peak height. A rectangular pulse with area A and width ε has height A/ε . In the exponential row, $a > 0$ gives decay; $a < 0$ gives growth. Use each row where its transform converges.

Time signal	Laplace transform
$\delta(t)$	1
$u_s(t)$, or 1 for $t \geq 0$	$1/s$
t	$1/s^2$
t^2	$2/s^3$
e^{-at}	$1/(s + a)$
$\sin(\omega t)$	$\omega/(s^2 + \omega^2)$
$\cos(\omega t)$	$s/(s^2 + \omega^2)$

Identify the family, match its parameter, and keep any scale factor. For $\sin(2t)$, use the sine row with $\omega = 2$ to obtain $2/(s^2 + 4)$. For a sum, transform each term separately:

$$\mathcal{L}\{1 + 2t\} = \frac{1}{s} + \frac{2}{s^2}.$$

This is linearity: coefficients remain attached to their terms.

Inverse transform: make a complete match

An inverse transform answers: which time signal gives this expression? Match both numerator and denominator. The sine row gives

$$\frac{1}{s^2 + 9} = \frac{1}{3} \frac{3}{s^2 + 3^2} \longleftrightarrow \frac{1}{3} \sin(3t).$$

Without the factor $1/3$, transforming back would give the wrong numerator. A cosine is also wrong here because its table numerator is s .

When one row is not enough: partial fractions

Split the expression into terms that match the table:

$$\frac{3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Multiplying both sides by the common denominator gives $3 = A(s+2) + B(s+1)$. Set $s = -1$: $3 = A$, so $A = 3$. Set $s = -2$: $3 = -B$, so $B = -3$. Each term now matches the exponential row:

$$\frac{3}{s+1} - \frac{3}{s+2} \longleftrightarrow 3e^{-t} - 3e^{-2t}.$$

The method is to rewrite first, then read the table. It avoids guessing the time signal from only part of the expression.

Derivatives, Initial Conditions, and Shifts

Derivatives: keep the starting condition

For the ordinary signals used here,

$$\begin{aligned}\mathcal{L}\{\dot{f}\} &= sF(s) - f(0), \\ \mathcal{L}\{\ddot{f}\} &= s^2F(s) - sf(0) - \dot{f}(0).\end{aligned}$$

Here $f(0)$ denotes the starting value. Do not drop an initial angle or velocity unless it is stated to be zero. For example, $q(0) = 0$ but $\dot{q}(0) = v_0$ gives $\mathcal{L}\{\ddot{q}\} = s^2Q - v_0$, not just s^2Q .

For $\dot{y} + 2y = 6u_s(t)$ with $y(0) = 1$, transform both sides and keep the initial value:

$$sY - 1 + 2Y = 6/s \quad \implies \quad Y = \frac{6/s + 1}{s + 2} = \frac{3}{s} - \frac{2}{s + 2}.$$

Read the table: $y(t) = 3 - 2e^{-2t}$ for $t \geq 0$. This starts at $3 - 2 = 1$, as required. In general, $Y = U/(s+2) + y(0)/(s+2)$: the initial-state term remains even without an input. For an integral starting at zero, $\mathcal{L}\{\int_0^t f(\lambda) d\lambda\} = F(s)/s$.

Two shifts: do not mix them up

Time delay moves the same waveform later, without changing its shape:

$$f(t - T)u_s(t - T) \quad \longleftrightarrow \quad e^{-Ts}F(s), \quad T \geq 0.$$

The signal is zero before T . Substituting $\lambda = t - T$ in the integral produces the factor e^{-sT} . For example, $2(t - 3)^2u_s(t - 3)$ has transform $4e^{-3s}/s^3$: first transform $2t^2$, then apply the delay.

Exponential weighting changes the envelope, not the starting time:

$$e^{at}f(t) \quad \longleftrightarrow \quad F(s - a).$$

This follows from $e^{at}e^{-st} = e^{-(s-a)t}$. Thus $te^{-5t} \leftrightarrow 1/(s + 5)^2$. Replace every s , including one in a numerator. When inverting, rewrite the numerator to match that shift:

$$\frac{2s + 5}{(s + 1)^2 + 4} = 2\frac{s + 1}{(s + 1)^2 + 2^2} + \frac{3}{2}\frac{2}{(s + 1)^2 + 2^2}.$$

The inverse is $2e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t)$. The rewrite $2s + 5 = 2(s + 1) + 3$ makes both table matches visible.

Initial and Final Values

Sometimes we need only the starting value or the settled value, not the whole curve. These two theorems provide shortcuts, but the final-value shortcut requires a check.

First, what is a pole?

For a rational expression, **cancel common factors first**; poles are the values of s that make the remaining denominator zero. For example, $1/(s+3)$ has one pole, $s = -3$. In $(s+1)/[(s+1)(s+3)]$, the factor $s+1$ cancels, so -1 is not a pole of the simplified expression.

The two shortcuts

Initial Value Theorem:

$$f(0^+) = \lim_{s \rightarrow +\infty} sF(s).$$

Use this for ordinary signals with a finite right-hand starting value and no impulse at the origin. The notation 0^+ means immediately after time begins, not one second later.

Final Value Theorem:

$$f(\infty) = \lim_{s \rightarrow 0} sF(s).$$

For the rational examples here, first simplify $sF(s)$ and check that *all its poles have negative real parts*. Only then evaluate the limit. Check sF , not the original F .

A temperature increase

Suppose $f(t)$ is the increase above the starting temperature:

$$\begin{aligned} F(s) &= \frac{10}{s(s+1)} && \text{poles: } 0, -1, \\ sF(s) &= \frac{10}{s+1} && \text{pole after cancellation: } -1. \end{aligned}$$

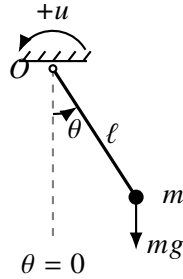
The initial value is $\lim_{s \rightarrow \infty} 10/(s+1) = 0$. The final-value check passes because the remaining pole is -1 . The final increase is $\lim_{s \rightarrow 0} 10/(s+1) = 10$ degrees Celsius. The pole at zero in F does not invalidate this result: it cancels in sF . This is a temperature *change*, not an absolute temperature.

Why the check matters: for $f(t) = \sin t$, the poles of $sF(s) = s/(s^2+1)$ are $\pm j$, whose real parts are zero, not negative. The signal never settles. Although $\lim_{s \rightarrow 0} s/(s^2+1) = 0$, zero is *not* its final value.

IVT and FVT can save an inverse-transform calculation when only the endpoints are needed. They do not tell us the complete curve, overshoot, or motion between those endpoints.

System modeling: the pendulum and its state

A model connects a physical input to the motion we want to predict. Consider a point mass m on a massless rigid rod of length ℓ , with a fixed pivot and no friction. The angle θ is measured from vertically downward. Positive angle and positive applied torque u are both counterclockwise. The measured output is the angle, $y = \theta$.



The weight acts downward. Its perpendicular lever arm is $\ell \sin \theta$. At the positive angle shown, gravity produces a clockwise, negative torque.

First, let it swing freely. The moment of inertia about the pivot is $m\ell^2$. The rod force has no moment about that pivot. With no applied torque, the torque balance is

$$m\ell^2\ddot{\theta} = -mg\ell \sin \theta.$$

Now add an input. A motor torque $u(t)$ contributes with the positive sign we defined, so

$$m\ell^2\ddot{\theta} = u - mg\ell \sin \theta, \quad \boxed{\ddot{\theta} = -\frac{g}{\ell} \sin \theta + \frac{u}{m\ell^2}.$$

Free motion means $u = 0$, not that gravity disappears. The sine term makes this model nonlinear; no small-angle approximation has been used yet.

Why keep two states? At the same angle, the pendulum may be moving in either direction. Angle alone cannot determine what happens next. Let $\omega = \dot{\theta}$ and choose

$$x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \quad \dot{x} = f(x, u) = \begin{bmatrix} \omega \\ -\frac{g}{\ell} \sin \theta + \frac{u}{m\ell^2} \end{bmatrix}, \quad y = \theta.$$

The first row gives the rate of angle; the second gives the rate of angular velocity. A state records enough present information to predict future motion when the model and future input are known.

System modeling: equilibrium and stability

An **equilibrium** is a state that remains constant under a constant input. A star marks its value. To find one, set every state rate to zero:

$$f(x^*, u^*) = 0.$$

For our free pendulum, $u^* = 0$. Apply this condition to *both* rows of the state equation:

$$\begin{aligned}\dot{\theta} = \omega = 0 &\implies \omega^* = 0, \\ \dot{\omega} = -\frac{g}{\ell} \sin \theta = 0 &\implies \sin \theta^* = 0.\end{aligned}$$

In the interval $0 \leq \theta < 2\pi$, this gives two states:

$$x_h^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{hanging}), \quad x_i^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad (\text{upright}).$$

The state itself need not be zero. Its *rate* must be zero. Upright is an equilibrium even though its angle is π rather than zero.

Passing through the bottom is not staying there

Suppose $\theta = 0$, $\omega = 1$ rad/s, and $u = 0$. Gravity has zero moment at that instant, so $\dot{\omega} = 0$. However, $\dot{\theta} = \omega = 1$ rad/s: the angle is changing. This is not an equilibrium because only one state rate is zero.

Finding an equilibrium answers whether the system can remain there exactly. **Stability** asks a different question: after a small initial disturbance, does it remain nearby? Hold the input at its equilibrium value while making this comparison.

- **Stable:** starting sufficiently close keeps the state close for all future time. Motion can continue; it need not stop.
- **Asymptotically stable:** stable, and the state approaches the equilibrium as time increases.
- **Unstable:** the stability condition fails. Arbitrarily small initial deviations can carry the state away from a fixed neighborhood.

Our free pendulum has no damping

Near hanging, a small disturbance produces small oscillations that do not settle: this equilibrium is stable but not asymptotically stable. Near upright, a small tilt can grow and the pendulum falls away: this equilibrium is unstable. Being balanced is not the same as being stable.

The next step is to build a simpler model for motion *near* one of these equilibria. We must identify which equilibrium before making that approximation.

System modeling: a local linear model

Linearization uses local slopes to approximate nonlinear equations near an equilibrium. Write actual quantities as equilibrium values plus small deviations:

$$x = x^* + \delta x, \quad u = u^* + \delta u, \quad y = y^* + \delta y.$$

Here $y^* = h(x^*, u^*)$ is the equilibrium output. A small deviation from upright means $\theta = \pi + \delta\theta$, not $\theta \approx 0$.

The first-order Taylor idea is “value at equilibrium plus local slope times change.” Since the equilibrium is constant,

$$\delta\dot{x} = f(x^* + \delta x, u^* + \delta u) \approx \underbrace{f(x^*, u^*)}_0 + A\delta x + B\delta u,$$

where A contains the state derivatives and B the input derivatives, evaluated at the equilibrium. Higher-order deviation terms are omitted.

Apply this to hanging: $\theta^* = 0$, $\omega^* = 0$, $u^* = 0$. The first equation is already linear. In the second, use $\sin(\delta\theta) \approx \delta\theta$:

$$\delta\dot{\theta} = \delta\omega, \quad \delta\ddot{\theta} \approx -\frac{g}{\ell}\delta\theta + \frac{1}{m\ell^2}\delta u.$$

Read the coefficient of each state and input, one row at a time:

$$\delta\dot{x} = A\delta x + B\delta u, \quad A = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/(m\ell^2) \end{bmatrix}.$$

The first row says angle changes at angular velocity. The second says gravity opposes a small angle deviation, while torque changes angular acceleration. For example, its state slope is $\partial\ddot{\theta}/\partial\theta = -(g/\ell)\cos\theta$, which becomes $-g/\ell$ at hanging.

The output determines C and D . Our measurement is $y = h(x, u) = \theta$. Substitute $y = y^* + \delta y$ and $\theta = \theta^* + \delta\theta$. Since $y^* = \theta^*$, subtracting the same baseline gives

$$\delta y = \delta\theta = C\delta x + D\delta u, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

C selects angle from the state. $D = 0$ means no direct input term in this measurement; torque still affects angle through the dynamics.

Keep the operating point and the approximation

These matrices describe small motion near hanging, not a full swing. For example, $\sin(0.1) \approx 0.0998$ is close to 0.1, but $\sin(1) \approx 0.8415$ differs appreciably from 1. If we shorten the notation to $\dot{x} = Ax + Bu$, $y = Cx + Du$, the variables still mean deviations from the chosen equilibrium.

Apply the linearization method

Use the same sequence for a new model: find the equilibrium input, define deviations, then evaluate the local derivatives. A first-order Taylor approximation keeps terms proportional to deviations; it does not mean that the dynamic system has only one state.

A scalar state equation

Let $\dot{x} = -x^2 + u$ and choose $x^* = 1$. A constant state requires $0 = -(x^*)^2 + u^*$, so $u^* = 1$. Write $\delta x = x - 1$ and $\delta u = u - 1$. The local coefficients are

$$A = \left. \frac{\partial f}{\partial x} \right|_* = -2x^* = -2, \quad B = \left. \frac{\partial f}{\partial u} \right|_* = 1.$$

Therefore $\delta \dot{x} \approx -2\delta x + \delta u$. To see what was omitted, substitute directly: $-(1 + \delta x)^2 + (1 + \delta u) = -2\delta x - (\delta x)^2 + \delta u$. The constant terms cancel; only the squared deviation is dropped.

A nonzero pendulum angle with damping

Suppose $\ddot{\theta} = -6 \sin \theta - 2\dot{\theta} + 3u$ and choose $\theta^* = \pi/6$. At equilibrium $\dot{\theta}^* = 0$ and

$$0 = -6 \sin(\pi/6) + 3u^* = -3 + 3u^* \implies u^* = 1.$$

Use $\delta x = [\theta - \pi/6, \dot{\theta}]^T$ and $\delta u = u - 1$. Each entry A_{ij} is the derivative of state equation i with respect to state j . Differentiate first, then use $\cos(\pi/6) = \sqrt{3}/2$:

$$A = \begin{bmatrix} 0 & 1 \\ -6 \cos(\pi/6) & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3\sqrt{3} & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The damping term is already linear, so its coefficient -2 stays. With angle output, $y^* = \pi/6$, $\delta y = y - \pi/6 = \delta x_1$, $C = [1 \ 0]$, and $D = 0$.

A nonlinear output equation

The output can have its own nonlinear rule. For $y = h(x, u) = x_1^2 + x_2 + 2u$ at $x^* = [1, 0]^T$, $u^* = 2$, first find $y^* = 1^2 + 0 + 2(2) = 5$. Then

$$C = \left. \frac{\partial h}{\partial x} \right|_* = [2 \ 1], \quad D = \left. \frac{\partial h}{\partial u} \right|_* = 2.$$

Subtract the output baseline: $\delta y = y - 5 \approx 2\delta x_1 + \delta x_2 + 2\delta u$. Here $D \neq 0$ because the measurement rule includes the input directly. This output calculation needs the specified operating point, not a new state equation.

System modeling: from state space to a transfer function

State space keeps the internal variables. A transfer function removes them to connect one chosen input directly to one chosen output. Start with the linear time-invariant deviation model

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Use capital letters for Laplace transforms. The derivative rule is

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0).$$

Assume $x(0) = 0$: the initial *deviation* is zero, so the system starts at its operating equilibrium. Taking transforms and eliminating the state gives

$$\begin{aligned} sX &= AX + BU, & Y &= CX + DU, \\ (sI - A)X &= BU, & X &= (sI - A)^{-1}BU, \\ Y &= [C(sI - A)^{-1}B + D]U. \end{aligned}$$

I is the identity matrix, and the inverse is used where it exists. For a single input and a single output, identify the factor multiplying U :

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D.$$

This ratio assumes zero initial state. If $x(0) \neq 0$, the initial motion adds $C(sI - A)^{-1}x(0)$ to the output; the total response is no longer just GU .

Use the two state rows to find $G(s)$

For the numerical model

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0,$$

the state equations are $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_1 + 2u$, with $y = x_1$. Under zero initial conditions,

$$sX_1 = X_2, \quad sX_2 = -X_1 + 2U, \quad Y = X_1.$$

Substitute $X_2 = sX_1$ into the second equation:

$$s^2 X_1 = -X_1 + 2U \implies (s^2 + 1)Y = 2U \implies G(s) = \frac{2}{s^2 + 1}.$$

This row-by-row calculation gives the same result as the matrix formula.

Use it to predict a response. For a step input with transform $U(s) = 0.05/s$,

$$Y(s) = G(s)U(s) = \frac{2}{s^2 + 1} \frac{0.05}{s} = \frac{0.1}{s(s^2 + 1)}.$$

The transfer function describes the selected system channel; U describes the applied input. Multiply them first, then use inverse Laplace if the time response is needed. Changing the measured output can change G , even when the underlying state dynamics stay the same.

Block diagrams: follow the signals

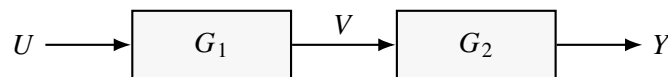
A block diagram records which signal goes where. Here we use scalar LTI models with zero initial conditions. U and Y are Laplace transforms; we omit (s) from the labels to keep the drawings readable.

A block transforms: $Y = GU$. If the block is the constant gain $G = 2$ and its input is $U = 3$, its output is $Y = 6$.

A summing point adds or subtracts: read the sign beside each incoming arrow. Desired angle 10° minus measured angle 8° gives error 2° . The signals must have compatible units.

A branch copies: if $U = 6$ enters a branch with two outgoing arrows, both carry 6, not 3. The same measurement can go to a controller and a display without splitting its value.

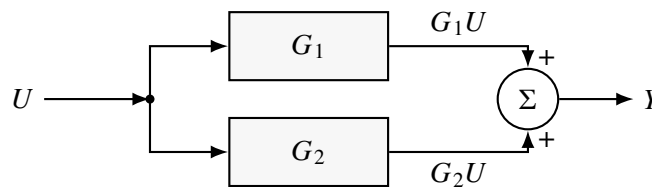
Series: one block feeds the next.



$$V = G_1 U, \quad Y = G_2 V \quad \Rightarrow \quad \boxed{Y/U = G_2 G_1.}$$

The second block acts on the first block's output, so the effects multiply. For scalar blocks the order of multiplication does not matter; do not assume this for matrix-valued blocks.

Parallel: both blocks receive the same input.

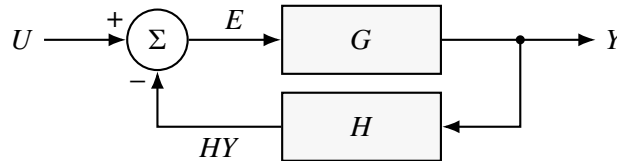


$$Y = G_1 U + G_2 U \quad \Rightarrow \quad \boxed{Y/U = G_1 + G_2.}$$

If the second incoming sign is minus, use $G_1 - G_2$ instead. For $G_1 = 2$, $G_2 = 3$, and $U = 1$, series gives 6, positive parallel gives 5, and subtracting the second parallel path gives -1 . The same blocks can produce different results because their connections are different.

Feedback: derive the equivalent block

Negative feedback subtracts the returned measurement. Follow the output Y through the return block H : the signal entering the minus side is HY , not simply Y . The resulting error E then passes through the forward block G .



Write the two signal equations first:

$$E = U - HY, \quad Y = GE.$$

Then substitute the error and collect the output terms:

$Y = G(U - HY)$	Substitute.
$Y = GU - GHY$	Expand.
$(1 + GH)Y = GU$	Collect all terms containing Y .

Therefore

$$\boxed{\frac{Y}{U} = \frac{G}{1 + GH}}.$$

The plus in the denominator comes from moving $-GHY$ to the left; it does not change the minus sign at the physical summing point.

A short numerical example

Let $G = 2$ and $H = 1/4$. Then

$$\frac{Y}{U} = \frac{2}{1 + 2(1/4)} = \frac{4}{3}.$$

The forward-path gain is 2 and the loop gain is $GH = 1/2$. Neither is the closed-loop gain $4/3$: the returning output also changes the error.

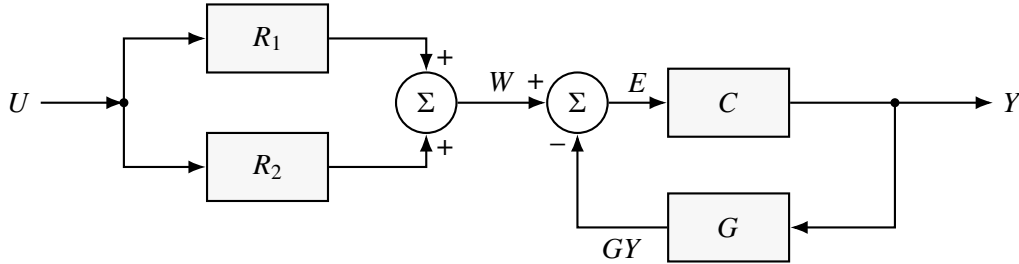
If the return sign is positive, use $E = U + HY$ instead. Rearranging $Y = G(U + HY)$ gives

$$(1 - GH)Y = GU, \quad \boxed{\frac{Y}{U} = \frac{G}{1 - GH}}.$$

The same numerical values now give $2/(1 - 1/2) = 4$. These constant-gain examples check the algebra, not dynamic stability. **A negative feedback sign alone does not guarantee stability.** For a dynamic model, stability depends on the resulting dynamics.

Two diagrams: read the return path

Example 1: parallel input paths, then a feedback loop. Follow the original connections before simplifying. Both R_1 and R_2 receive U ; the returned measurement comes from Y .



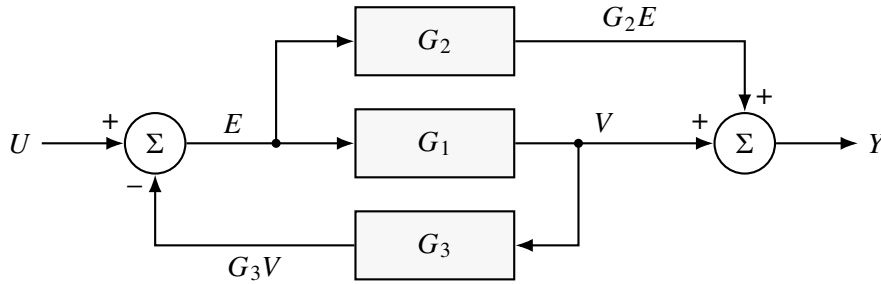
$$W = (R_1 + R_2)U, \quad E = W - GY, \quad Y = CE.$$

The feedback encloses C , not the two input paths. Combine the parallel paths, then reduce the feedback subsystem:

$$\boxed{\frac{Y}{U} = \frac{(R_1 + R_2)C}{1 + CG}}.$$

With $R_1 = 1$, $R_2 = 2$, $C = 2$, $G = 1$, the gain is $(1 + 2)2/(1 + 2 \cdot 1) = 2$. Here C names a transfer block, not a state-space matrix.

Example 2: feedback is taken before the final sum. The error E feeds both forward blocks. Only $V = G_1 E$ is returned through G_3 ; the feedback does not measure Y .



$$E = U - G_3V, \quad V = G_1E, \quad Y = V + G_2E.$$

Thus $E = U/(1 + G_1G_3)$ and

$$\boxed{\frac{Y}{U} = \frac{G_1 + G_2}{1 + G_1G_3}}.$$

Remember: the takeoff point decides which blocks are inside the feedback loop. Here the denominator contains G_1G_3 , not $(G_1 + G_2)G_3$.

Drone lab: from motion to a local model

The activity connects the earlier topics in one example. We control vertical motion using a motor-speed adjustment; the output of interest is height. The model is illustrative, not a calibrated commercial-drone model.

Symbol	Physical meaning
z	Height in metres; upward is positive.
$v = \dot{z}$	Vertical velocity in m/s.
q	Applied motor speed relative to hover speed; dimensionless.
d	Wind acceleration in m/s ² .

Read the nonlinear equation physically

$$\dot{z} = v, \quad \dot{v} = 10(q^2 - 1) - v + d.$$

The motor term $10(q^2 - 1)$ is zero at the hover setting $q = 1$. The term $-v$ opposes motion. The numerical coefficients carry the appropriate SI units; the acceleration is in m/s².

Hover speed is not a stop button

Suppose the drone is moving upward and we set $q = 1$, with no wind. Then $\dot{v} = -v$: upward velocity gradually decreases. But $\dot{z} = v$ is still positive, so height keeps increasing while the drone slows. The motor setting does not instantly reset velocity or height.

Linearize by expanding about hover

Write the applied adjustment as $\delta q = q - 1$. With no wind,

$$\begin{aligned} \dot{v} &= 10[(1 + \delta q)^2 - 1] - v \\ &= 20\delta q + 10(\delta q)^2 - v \\ &\approx 20\delta q - v. \end{aligned}$$

We drop the squared adjustment, not the velocity term. At $\delta q = 0.05$, the neglected acceleration is $10(0.05)^2 = 0.025$ m/s². At $\delta q = 0.30$, it is 0.9 m/s². Larger adjustments can make the approximation less accurate.

What the linear model retains

It still has motion and memory: velocity changes over time, and height accumulates velocity. Linearization simplifies the governing rule; it does not make the drone respond instantly.

Drone lab: motor, velocity and height blocks

Define the changes before taking Laplace transforms

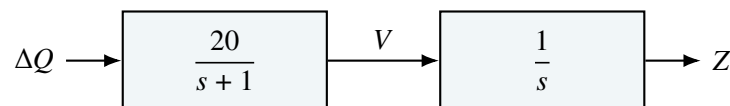
The starting hover height is 1 m. Use height deviation $\zeta = z - 1$, with initial values $\zeta(0) = 0$ and $v(0) = 0$. Let Z , V , and ΔQ be the transforms of ζ , v , and δq . **Z means transformed height change, not absolute height.**

For the local model with no wind,

$$\begin{aligned}\dot{v} + v &= 20\delta q &\implies & sV + V = 20\Delta Q, \\ \dot{\zeta} &= v &\implies & sZ = V.\end{aligned}$$

The initial terms vanish because of the stated starting conditions. Dividing output by input gives

$$\frac{V}{\Delta Q} = \frac{20}{s+1}, \quad \frac{Z}{V} = \frac{1}{s}.$$



The first block predicts velocity from the motor adjustment. The second integrates velocity to get height change. They are in series, so the complete plant is

$$G_z(s) = \frac{Z}{\Delta Q} = \frac{20}{s(s+1)}.$$

Changing the output changes the transfer function

$20/(s+1)$ is a motor-to-*velocity* model. $20/[s(s+1)]$ is a motor-to-*height* model. A transfer function is not fully specified without naming its input and output.

Use the input to predict the output

For a small step adjustment $\delta q(t) = a u_s(t)$, the transform is $\Delta Q = a/s$. Therefore

$$Z(s) = G_z(s)\Delta Q(s) = \frac{20a}{s^2(s+1)}.$$

The extra $1/s$ here comes from the *step input*. The $1/s$ already inside G_z comes from velocity accumulating into height. The plant and the input describe different parts of the experiment.

Next: close the loop

So far, an adjustment is supplied to the drone. Feedback instead calculates that adjustment from desired height minus measured height, and updates it as the drone moves.

Drone lab: feedback and actuator limits

For desired absolute height r and measured height z , a proportional controller requests

$$e = r - z, \quad \delta q_{\text{req}} = Ke, \quad K > 0.$$

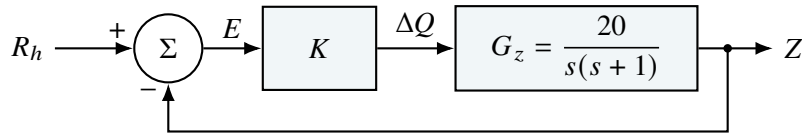
For example, $r = 1.2$, $z = 1.0$ and $K = 0.1 \text{ m}^{-1}$ give a request of 0.02. Adding the hover baseline gives $q = 1.02$: speed up the motors.

Use the same baseline on both sides of the comparison

Define $r_h = r - 1$ and $\zeta = z - 1$. Then

$$e = r - z = (1 + r_h) - (1 + \zeta) = r_h - \zeta.$$

For small changes, no wind, zero initial deviations and no saturation, the Laplace-domain diagram is



Read the loop, substitute, then collect the output terms:

$$Z = G_z K (R_h - Z) \implies (1 + KG_z)Z = KG_z R_h.$$

Hence

$$\boxed{\frac{Z}{R_h} = \frac{KG_z}{1 + KG_z} = \frac{20K}{s^2 + s + 20K}}.$$

This closed-loop model predicts height change from a desired height change. The plant alone predicts height change from a motor adjustment.

A request is not always the applied input

The actuator applies the limited setting

$$q = \text{clip}(1 + \delta q_{\text{req}}, 0.7, 1.3), \quad \delta q = q - 1.$$

Here `clip` keeps the setting within its allowed range: anything above 1.3 becomes 1.3, and anything below 0.7 becomes 0.7. Once clipping occurs, $\delta q \neq Ke$ in general, so the linear closed-loop formula no longer describes that saturated motion. A larger K asks for a stronger correction; it does not guarantee a better flight.

Judge both tracking and effort. Check how closely height follows the target and how much the applied motor setting changes. A cost such as $J_q = \int_0^T (q - 1)^2 dt$ measures adjustment effort, not battery energy. Since q is dimensionless, J_q has units of seconds. Compare designs over the same conditions and time interval.

Practice: connect the ideas

Use the transform table. Show the main step and one sentence explaining what your answer means. These questions support revision; they are not a separate statement of exam coverage.

- 1. Linearity and state.** Is $y = u^2$ linear? Test it by doubling u . A motor receives zero voltage but is still spinning. Why is its present voltage alone insufficient to predict what happens next?
- 2. Use the table and keep the starting state.** Find $\mathcal{L}\{3t\}$. If $q(0) = 0$ and $\dot{q}(0) = 2$, find $\mathcal{L}\{\ddot{q}\}$ in terms of $Q(s)$. Which term would be lost by using only s^2Q ?
- 3. Inverse transform.** Find the time signal corresponding to

$$F(s) = \frac{2}{(s+1)(s+3)}.$$

Use partial fractions, then match the exponential row.

- 4. Initial and final values.** For $F(s) = 6/[s(s+3)]$, find $f(0^+)$ and $f(\infty)$. Show which poles must be checked for the final value. Would the same final-value method work for $1/(s^2+1)$?
- 5. Build a local model yourself.** A pendulum obeys $\ddot{\theta} = -4\sin\theta + 2u$, with output $y = \theta$. Linearize at $(\theta^*, \dot{\theta}^*, u^*) = (\pi, 0, 0)$. Define the state, input and output deviations, and derive A, B, C, D . Is the local output the absolute angle or the change from upright?
- 6. Read negative feedback.** A loop has forward gain $G = 2$ and return gain $H = 0.5$. Write the error and output equations, then calculate Y/U . Why is Y/U not simply 2?
- 7. Check the local drone approximation.** At $\delta q = 0.10$, $v = 0.50$ m/s and no wind, calculate the nonlinear and linearized accelerations. Which omitted term accounts for the difference? Which block turns the velocity prediction into height change?
- 8. Request versus applied input.** For a written calculation, take $r = 1.3$ m, $z = 1.0$ m and $K = 2$ m⁻¹. Find δq_{req} , the applied q , and $\delta q = q - 1$, using $0.7 \leq q \leq 1.3$. Can the unsaturated linear feedback equation still be used for this applied input?

Explained answers: signals and Laplace

1. Doubling is a useful first test

For $y = u^2$, doubling u gives $(2u)^2 = 4u^2$, rather than twice the original output. Homogeneity fails, so the map is nonlinear.

A stopped motor and a spinning motor can have the same zero voltage. Their future motion differs because their states differ. Angular velocity is therefore part of the information needed to predict the motion.

2. The coefficient and initial velocity both matter

The ramp row gives $\mathcal{L}\{t\} = 1/s^2$. By linearity,

$$\mathcal{L}\{3t\} = \frac{3}{s^2}.$$

For the acceleration, begin with the complete derivative rule:

$$\mathcal{L}\{\ddot{q}\} = s^2 Q - sq(0) - \dot{q}(0) = s^2 Q - 2.$$

The missing term would be -2 . Zero initial position is not the same as zero initial state: the system is already moving.

3. Split the fraction before using the table

Set

$$\frac{2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}.$$

Multiplying through gives $2 = A(s+3) + B(s+1)$. At $s = -1$, $2 = 2A$, so $A = 1$. At $s = -3$, $2 = -2B$, so $B = -1$. Thus

$$F(s) = \frac{1}{s+1} - \frac{1}{s+3} \implies \boxed{f(t) = e^{-t} - e^{-3t}, \quad t \geq 0.}$$

The two denominator constants become two different exponential decay rates.

4. Check the expression used by the theorem

First multiply and simplify:

$$sF(s) = \frac{6}{s+3}.$$

Its only pole is -3 , so the final-value check passes:

$$f(0^+) = \lim_{s \rightarrow +\infty} \frac{6}{s+3} = 0, \quad f(\infty) = \lim_{s \rightarrow 0} \frac{6}{s+3} = 2.$$

For $1/(s^2 + 1)$, sF has poles at $\pm j$, on the imaginary axis. The signal is $\sin t$ and never settles to a constant. The algebraic limit of zero is not a valid final value.

Explained answers: models and feedback

5. Measure small changes from upright

Choose $x = [\theta \ \dot{\theta}]^T$ and $x^* = [\pi \ 0]^T$. Then $\delta x_1 = \theta - \pi$, $\delta x_2 = \dot{\theta}$, $\delta u = u$, and $\delta y = y - \pi = \delta x_1$. Substitute and use $\sin(\pi + \delta x_1) \approx -\delta x_1$:

$$\delta \dot{x}_1 = \delta x_2, \quad \delta \dot{x}_2 \approx 4\delta x_1 + 2\delta u.$$

Therefore

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C = [1 \ 0], \quad D = 0.$$

The output is the change from upright. Recover absolute angle with $y = \pi + \delta y$. With zero input deviation, a positive angle deviation has a positive acceleration: the upright equilibrium does not provide a restoring effect.

6. The return path changes the input to the block

The diagram gives $E = U - 0.5Y$ and $Y = 2E$. Substituting,

$$Y = 2(U - 0.5Y) = 2U - Y \implies 2Y = 2U \implies \boxed{Y/U = 1.}$$

The gain 2 maps error to output, not external input directly to output. Feedback makes the error depend on that same output.

7. The quadratic term is the difference

Here $q = 1.10$. The nonlinear and local predictions are

$$\dot{v}_{\text{nonlinear}} = 10(1.10^2 - 1) - 0.50 = 1.60 \text{ m/s}^2, \quad \dot{v}_{\text{linear}} = 20(0.10) - 0.50 = 1.50 \text{ m/s}^2.$$

Their difference is $10(\delta q)^2 = 0.10 \text{ m/s}^2$. The block $1/s$ accumulates velocity into height change, with zero initial height deviation. A velocity model alone is not a height model.

8. Clipping limits what reaches the plant

$$e = 1.3 - 1.0 = 0.3, \quad \delta q_{\text{req}} = 2(0.3) = 0.6,$$

$$q = \text{clip}(1.6, 0.7, 1.3) = 1.3, \quad \boxed{\delta q = q - 1 = 0.3.}$$

The applied adjustment is 0.3, not 0.6. Thus $\delta q = Ke$ fails: the unsaturated linear feedback equation does not describe this motion. $K = 2$ is a written exercise value, not a gain setting for the activity interface.

More examples: [Laplace](#), [modeling](#), [block diagrams](#), [drone activity](#). The [original 15-question diagnostic answers](#) remain on their own review page.