Supplementary Materials: Analytic Combined IMU Integration (ACI^2) For Visual Inertial Navigation

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Contents

1 Hamilton Quaternion

In this section, we would like to give the basic operation of Hamilton Quaternion [1] and show that the right Jacobians for Hamilton Quaternion is equivalent to the left Jacobians of JPL Quaternion and the right Jacobians of $\mathbf{SO}(3)$.

1.1 Hamilton Quaternion Operation

The Hamilton quaternion is used for rotation representation. A quaternion $\bar{q}$, with rotation axis $\frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|}$ and rotation angle $\|\boldsymbol{\phi}\|$, can be written as:

$$\bar{q} = \begin{bmatrix} q_w \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} \cos\frac{\|\boldsymbol{\phi}\|}{2} \\ \sin\frac{\|\boldsymbol{\phi}\|}{2} \cdot \frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|} \end{bmatrix} \tag{1}$$

The quaternion operation $\otimes$ can be defined as:

$$\bar{q} \otimes \bar{p} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} \otimes \begin{bmatrix} p_w \\ \mathbf{p}_v \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} q_w & -\mathbf{q}_v^\top \\ \mathbf{q}_v & q_w \mathbf{I}_3 + \lfloor \mathbf{q}_v \rfloor \end{bmatrix} \begin{bmatrix} p_w \\ \mathbf{p}_v \end{bmatrix} = \mathcal{L}(\bar{q})\bar{p} \tag{3}$$

$$= \begin{bmatrix} p_w & -\mathbf{p}_v^\top \\ \mathbf{p}_v & p_w \mathbf{I}_3 - \lfloor \mathbf{p}_v \rfloor \end{bmatrix} \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \mathcal{R}(\bar{p})\bar{q} \tag{4}$$

where $\mathcal{R}(\cdot)$ and $\mathcal{L}(\cdot)$ are defined as the right and left multiplication, respectively.

1.2 Right Jacobians for Hamilton Quaternion

We define the quaternion error states $\delta\boldsymbol{\phi}$ and the quaternion tangent error states $\delta\boldsymbol{\theta}$ as:

$$\bar{q}(\boldsymbol{\phi} + \delta\boldsymbol{\theta}) = \bar{q}(\boldsymbol{\phi}) \otimes \bar{q}(\delta\boldsymbol{\phi}) = \bar{q}(\boldsymbol{\phi}) \otimes \bar{q}(\mathbf{J}_r^{(H)} \delta\boldsymbol{\theta}) \tag{5}$$

$$\Rightarrow \bar{q}(\delta\boldsymbol{\phi}) = \bar{q}^{-1}(\boldsymbol{\phi}) \otimes \bar{q}(\boldsymbol{\phi} + \delta\boldsymbol{\theta}) = \bar{q}(\mathbf{J}_r^{(H)} \delta\boldsymbol{\theta}) \tag{6}$$

With small angle approximation, we can derive (6) as:

$$\begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\phi} \end{bmatrix} \simeq \begin{bmatrix} \cos\frac{1}{2}\|\boldsymbol{\phi}\| & \sin\frac{1}{2}\|\boldsymbol{\phi}\| \cdot \frac{\boldsymbol{\phi}^\top}{\|\boldsymbol{\phi}\|} \\ -\sin\frac{1}{2}\|\boldsymbol{\phi}\| \cdot \frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|} & \cos\frac{1}{2}\|\boldsymbol{\phi}\|\mathbf{I}_3 - \sin\frac{1}{2}\|\boldsymbol{\phi}\|\lfloor\frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|}\rfloor \end{bmatrix} \begin{bmatrix} \cos\frac{1}{2}\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\| \\ \sin\frac{1}{2}\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\| \cdot \frac{\boldsymbol{\phi}+\delta\boldsymbol{\theta}}{\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\|} \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \cdot \mathbf{k}^\top \\ -\sin\frac{\phi}{2} \cdot \mathbf{k} & \cos\frac{\phi}{2} \cdot \mathbf{I}_3 - \sin\frac{\phi}{2}\lfloor\mathbf{k}\rfloor \end{bmatrix} \begin{bmatrix} \cos\frac{1}{2}\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\| \\ \sin\frac{1}{2}\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\| \cdot \frac{\boldsymbol{\phi}+\delta\boldsymbol{\theta}}{\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\|} \end{bmatrix} \tag{8}$$

$$\simeq \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \cdot \mathbf{k}^\top \\ -\sin\frac{\phi}{2} \cdot \mathbf{k} & \cos\frac{\phi}{2} \cdot \mathbf{I}_3 - \sin\frac{\phi}{2}\lfloor\mathbf{k}\rfloor \end{bmatrix} \begin{bmatrix} \cos\frac{\phi}{2} - \frac{1}{2}\sin\frac{\phi}{2} \cdot \mathbf{k}^\top\delta\boldsymbol{\theta} \\ \sin\frac{\phi}{2} \cdot \mathbf{k} + \left(\frac{1}{2}\cos\frac{\phi}{2} \cdot \mathbf{k}\mathbf{k}^\top + \sin\frac{\phi}{2} \cdot \frac{\mathbf{I}_3 - \mathbf{k}\mathbf{k}^\top}{\phi}\right)\delta\boldsymbol{\theta} \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 1 \\ \mathbf{0}_{3\times1} \end{bmatrix} + \begin{bmatrix} \sin^2\frac{\phi}{2} \cdot \mathbf{k}^\top\frac{\mathbf{I}_3 - \mathbf{k}\mathbf{k}^\top}{\phi} \\ \frac{1}{2}\sin^2\frac{\phi}{2} \cdot \mathbf{k}\mathbf{k}^\top + \left(\cos\frac{\phi}{2} \cdot \mathbf{I}_3 - \sin\frac{\phi}{2} \cdot \lfloor\mathbf{k}\rfloor\right)\left(\frac{1}{2}\cos\frac{\phi}{2} \cdot \mathbf{k}\mathbf{k}^\top + \sin\frac{\phi}{2} \cdot \frac{\mathbf{I}_3 - \mathbf{k}\mathbf{k}^\top}{\phi}\right) \end{bmatrix}\delta\boldsymbol{\theta} \tag{10}$$

$$= \begin{bmatrix} 1 \\ \mathbf{0}_{3\times1} \end{bmatrix} + \begin{bmatrix} \sin^2\frac{\phi}{2} \cdot \mathbf{k}^\top\frac{\mathbf{I}_3 - \mathbf{k}\mathbf{k}^\top}{\phi} \\ \frac{1}{2}\mathbf{k}\mathbf{k}^\top + \frac{1}{2}\sin\frac{\phi}{2} \cdot \frac{\mathbf{I}_3 - \mathbf{k}\mathbf{k}^\top}{\phi} - \sin^2\frac{\phi}{2} \cdot \frac{\lfloor\mathbf{k}\rfloor}{\phi} \end{bmatrix}\delta\boldsymbol{\theta} \tag{11}$$

$$= \begin{bmatrix} 1 \\ \mathbf{0}_{3\times1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2}\left(\mathbf{I}_3 - (1-\cos\phi)\frac{\lfloor\mathbf{k}\rfloor}{\phi} + \frac{\phi - \sin\phi}{\phi}\lfloor\mathbf{k}\rfloor^2\right) \end{bmatrix}\delta\boldsymbol{\theta} \tag{12}$$

where we have defined $\phi = \|\boldsymbol{\phi}\|$ and $\mathbf{k} = \frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|}$. Therefore, we can get the right Jacobians as:

$$\mathbf{J}_r^{(H)}(\boldsymbol{\phi}) = \frac{\partial\delta\boldsymbol{\phi}}{\partial\delta\boldsymbol{\theta}} = \mathbf{I}_3 - \frac{1-\cos\phi}{\phi}\lfloor\mathbf{k}\rfloor + \frac{\phi-\sin\phi}{\phi}\lfloor\mathbf{k}\rfloor^2 \tag{13}$$

1.3 Left Jacobians for JPL Quaternion

We define the quaternion error states $\delta\boldsymbol{\phi}$ and the quaternion tangent error states $\delta\boldsymbol{\theta}$ as:

$$\bar{q}_J(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \bar{q}_J(\delta\boldsymbol{\phi}) \otimes \bar{q}_J(\boldsymbol{\phi}) = \bar{q}_J(\mathbf{J}_l^{(J)}\delta\boldsymbol{\theta}) \otimes \bar{q}_J(\boldsymbol{\phi}) \tag{14}$$

$$\Rightarrow \bar{q}_J(\delta\boldsymbol{\phi}) = \bar{q}_J(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) \otimes \bar{q}_J^{-1}(\boldsymbol{\phi}) = \bar{q}_J(\mathbf{J}_l^{(J)}\delta\boldsymbol{\theta}) \tag{15}$$

where $\bar{q}_J$ denotes JPL quaternion [2]. Based on the JPL quaternion operation defined in [2], we can follow the same procedure (7) to compute $\mathbf{J}_l^{(J)}$.

1.4 Right Jabcobians for $\mathbf{SO}(3)$

Currently, we only find a proof for right Jacobians of $\mathbf{SO}(3)$ presented in [3], which is referred by [4]. However, this proof is not easy to find and might be a little abstract for researchers in SLAM community. In this paper, we would provide a proof for right Jacobians $\mathbf{J}_r^{(S)}$ in a more intuitive way.

We want to show that:

$$\mathbf{R}(\delta\boldsymbol{\phi}) = \mathbf{R}(\boldsymbol{\phi})^{-1}\mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) \tag{16}$$

with $\mathbf{J}_r^{(S)}(\boldsymbol{\phi}) = \frac{\partial\delta\boldsymbol{\phi}}{\partial\delta\boldsymbol{\theta}} = \mathbf{I}_3 - \frac{1-\cos\phi}{\phi}\lfloor\mathbf{k}\rfloor + \frac{\phi-\sin\phi}{\phi}\lfloor\mathbf{k}\rfloor^2$. For simplicity, we only need to prove the following:

$$\mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) \tag{17}$$

We first introduce some important properties for skew-symmetric matrix:

$$\lfloor \mathbf{k} \rfloor^2 = \mathbf{k}\mathbf{k}^\top - \mathbf{I}_3 \tag{18}$$

$$\lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor = \delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}\delta\boldsymbol{\theta}^\top \tag{19}$$

$$\lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor = \lfloor \lfloor \mathbf{k} \rfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\boldsymbol{k}^\top + \mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor \tag{20}$$

$$\lfloor \mathbf{k} \rfloor \lfloor \delta\boldsymbol{\theta} \rfloor = \delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{I}_3 \tag{21}$$

$$\lfloor \mathbf{k} \rfloor \lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top - \lfloor \mathbf{k} \rfloor \mathbf{k}\delta\boldsymbol{\theta}^\top = \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top \tag{22}$$

$$\lfloor \mathbf{k} \rfloor^2 \lfloor \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor (\delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{I}_3) = \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}^\top \delta\boldsymbol{\theta} \lfloor \mathbf{k} \rfloor \tag{23}$$

$$\lfloor \mathbf{k} \rfloor^2 \lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor \lfloor \mathbf{k} \rfloor (\delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}\delta\boldsymbol{\theta}^\top) = \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta}\mathbf{k}^\top \tag{24}$$

$$\lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor \lfloor \mathbf{k} \rfloor^2 = (\delta\boldsymbol{\theta}\mathbf{k}^\top - \mathbf{k}\delta\boldsymbol{\theta}^\top) \lfloor \mathbf{k} \rfloor \lfloor \mathbf{k} \rfloor = -\mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor^2 \tag{25}$$

$$\lfloor \mathbf{k} \rfloor^2 \lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor \lfloor \mathbf{k} \rfloor (\lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top + \mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor) = -\lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top \tag{26}$$

$$\lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor \lfloor \mathbf{k} \rfloor^2 = (\lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top + \mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor) \lfloor \mathbf{k} \rfloor \lfloor \mathbf{k} \rfloor = -\mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor \tag{27}$$

$$\lfloor \mathbf{k} \rfloor \lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor = \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta}\mathbf{k}^\top \tag{28}$$

$$\lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor \lfloor \mathbf{k} \rfloor = \mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor^2 \tag{29}$$

$$\lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta}\mathbf{k}^\top = \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{k}\mathbf{k}^\top - \delta\boldsymbol{\theta}\mathbf{k}^\top \tag{30}$$

$$\mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor^2 = \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{k}\mathbf{k}^\top - \mathbf{k}\delta\boldsymbol{\theta}^\top \mathbf{I}_3 \tag{31}$$

For the right hand side of Eq(17), we can substitute the right Jacobians as follows:

$$\begin{aligned}
\mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) &\simeq \mathbf{R}(\boldsymbol{\phi})(\mathbf{I}_3 + \lfloor \mathbf{J}_r^{(S)}\delta\boldsymbol{\theta} \rfloor) && (32)\\
&= \mathbf{R}(\boldsymbol{\phi}) + \mathbf{R}(\boldsymbol{\phi}) \lfloor \left(\mathbf{I}_3 - \frac{1-\cos\phi}{\phi} \lfloor \mathbf{k} \rfloor + \frac{\phi - \sin\phi}{\phi} \lfloor \mathbf{k} \rfloor^2 \right) \delta\boldsymbol{\theta} \rfloor && (33)\\
&= \mathbf{R}(\boldsymbol{\phi}) + \lfloor \delta\boldsymbol{\theta} \rfloor \\
&- \frac{1-\cos\phi}{\phi} \lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor + \frac{\phi - \sin\phi}{\phi} \lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor \\
&+ \sin\phi \cdot \lfloor \mathbf{k} \rfloor \lfloor \delta\boldsymbol{\theta} \rfloor - \frac{\sin\phi \cdot (1-\cos\phi)}{\phi} \lfloor \mathbf{k} \rfloor \lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor \\
&+ \frac{\sin\phi \cdot (\phi - \sin\phi)}{\phi} \lfloor \mathbf{k} \rfloor \lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor + (1-\cos\phi) \lfloor \mathbf{k} \rfloor^2 \lfloor \delta\boldsymbol{\theta} \rfloor \\
&- \frac{(1-\cos\phi)^2}{\phi} \lfloor \mathbf{k} \rfloor^2 \lfloor \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta} \rfloor + \frac{(1-\cos\phi)(\phi - \sin\phi)}{\phi} \lfloor \mathbf{k} \rfloor^2 \lfloor \lfloor \mathbf{k} \rfloor^2 \delta\boldsymbol{\theta} \rfloor && (34)\\
&= \mathbf{R}(\boldsymbol{\phi}) + \lfloor \delta\boldsymbol{\theta} \rfloor \\
&+ \frac{1-\cos\phi}{\phi} \delta\boldsymbol{\theta}\mathbf{k}^\top + \frac{1-\cos\phi}{\phi} \mathbf{k}\delta\boldsymbol{\theta}^\top \\
&+ (1 - \frac{\sin\phi}{\phi}) \lfloor \mathbf{k} \rfloor \delta\boldsymbol{\theta}\mathbf{k}^\top - \sin\phi \cdot \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{I}_3 \\
&+ \cos\phi \cdot \mathbf{k}^\top \delta\boldsymbol{\theta} \lfloor \mathbf{k} \rfloor - \frac{\sin\phi}{\phi} \mathbf{k}\delta\boldsymbol{\theta}^\top \lfloor \mathbf{k} \rfloor + (\sin\phi - 2\frac{1-\cos\phi}{\phi}) \mathbf{k}^\top \delta\boldsymbol{\theta}\mathbf{k}\mathbf{k}^\top && (35)
\end{aligned}$$

For the left hand side of Eq(17) we have:

$$\mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \mathbf{I}_3 + \sin\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\| \cdot \lfloor \frac{\boldsymbol{\phi}+\delta\boldsymbol{\theta}}{\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\|} \rfloor + (1-\cos\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\|) \lfloor \frac{\boldsymbol{\phi}+\delta\boldsymbol{\theta}}{\|\boldsymbol{\phi}+\delta\boldsymbol{\theta}\|} \rfloor^2 \tag{36}$$

$$\begin{aligned}
&\simeq \mathbf{I}_3 + \sin\phi\lfloor\mathbf{k}\rfloor \\
&+ (1-\cos\phi)\lfloor\mathbf{k}\rfloor^2 - \frac{\sin\phi}{\phi}\lfloor\lfloor\mathbf{k}\rfloor^2\delta\boldsymbol{\theta}\rfloor \\
&+ \sin\phi\cdot\mathbf{k}^\top\delta\boldsymbol{\theta}\lfloor\mathbf{k}\rfloor^2 + \cos\phi\cdot\mathbf{k}^\top\delta\boldsymbol{\theta}\lfloor\mathbf{k}\rfloor + (1-\cos\phi)\lfloor\mathbf{k}\rfloor^2 \\
&- (1-\cos\phi)(\frac{\lfloor\mathbf{k}\rfloor\lfloor\lfloor\mathbf{k}\rfloor^2\delta\boldsymbol{\theta}\rfloor}{\phi} + \frac{\lfloor\lfloor\mathbf{k}\rfloor^2\delta\boldsymbol{\theta}\rfloor\lfloor\mathbf{k}\rfloor}{\phi})
\end{aligned} \tag{37}$$

$$\begin{aligned}
&= \mathbf{R}(\boldsymbol{\phi}) \\
&+ \frac{1-\cos\phi}{\phi}(\delta\boldsymbol{\theta}\mathbf{k}^\top + \mathbf{k}\delta\boldsymbol{\theta}^\top) \\
&- \frac{\sin\phi}{\phi}\lfloor\mathbf{k}\rfloor\delta\boldsymbol{\theta}\mathbf{k}^\top + \cos\phi\cdot\mathbf{k}^\top\delta\boldsymbol{\theta}\lfloor\mathbf{k}\rfloor \\
&- \frac{\sin\phi}{\phi}\mathbf{k}\delta\boldsymbol{\theta}^\top\lfloor\mathbf{k}\rfloor - \sin\phi\cdot\mathbf{k}^\top\delta\boldsymbol{\theta}\mathbf{I}_3 \\
&+ (\sin\phi - 2\frac{1-\cos\phi}{\phi})\mathbf{k}^\top\delta\boldsymbol{\theta}\mathbf{k}\mathbf{k}^\top
\end{aligned} \tag{38}$$

Then we can find:

$$\mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) - \mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \lfloor\delta\boldsymbol{\theta}\rfloor + \lfloor\mathbf{k}\rfloor\delta\boldsymbol{\theta}\mathbf{k}^\top \tag{39}$$

where we have used the following equality: for $\forall\mathbf{k}$, we have:

$$(\lfloor\delta\boldsymbol{\theta}\rfloor + \lfloor\mathbf{k}\rfloor\delta\boldsymbol{\theta}\mathbf{k}^\top)\mathbf{k} = \lfloor\delta\boldsymbol{\theta}\rfloor\mathbf{k} + \lfloor\mathbf{k}\rfloor\delta\boldsymbol{\theta} = \mathbf{0} \tag{40}$$

$$\Rightarrow \Big(\mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) - \mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta})\Big)\cdot\mathbf{k} = \mathbf{0} \tag{41}$$

$$\Rightarrow \Big(\mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) - \mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\delta\boldsymbol{\phi})\Big)\cdot\mathbf{k} = \mathbf{0} \tag{42}$$

Therefore, we can have:

$$\mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\delta\boldsymbol{\phi}) = \mathbf{R}(\boldsymbol{\phi}+\delta\boldsymbol{\theta}) = \mathbf{R}(\boldsymbol{\phi})\mathbf{R}(\mathbf{J}_r^{(S)}\delta\boldsymbol{\theta}) \tag{43}$$

We get the proof.

2 ACI^2 MODEL 1

In this section, we present the derivations for analytic combined IMU integration (ACI^2) MODEL 1. MODEL 1 assumes the acceleration measurement is constant between consecutive sampling intervals.

2.1 Integration

The synchronized time line for IMU readings and vision sensors can shown in Fig. 1.

The time interval from k to time j can be further divided as Fig. 2.

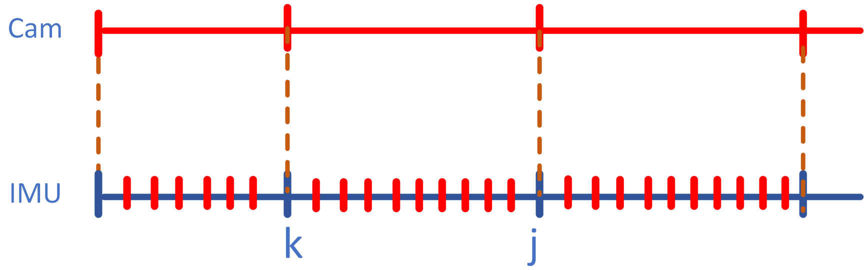

Figure 1: Aligned camera and IMU time line.

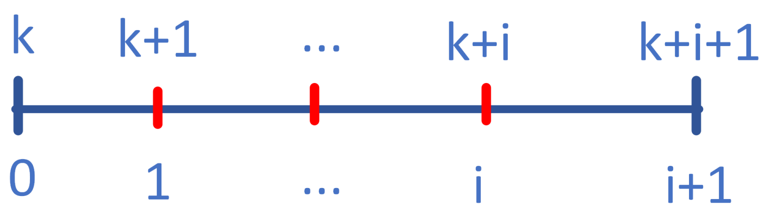

Figure 2: Time stamp notation for IMU integration

For simplicity, we also define the following notations:

$$j \triangleq k + i + 1 \tag{44}$$

$$\delta t_i \triangleq t_{k+i+1} - t_{k+i} \tag{45}$$

$$\Delta t_i \triangleq t_{k+i} - t_k \tag{46}$$

where $i \geq 0$. $\boldsymbol{\omega}_i$ and $\mathbf{a}_i$ represent the angular and linear velocity in the interval i (between t_{k+i} and t_{k+i+1}) and they are assumed constant to simplify the derivation. We can use IMU reading at t_{k+i} for $\boldsymbol{\omega}_i$ and $\mathbf{a}_i$ as:

$$\boldsymbol{\omega}_{mi} \triangleq {}^{I_{k+i}}\boldsymbol{\omega} \tag{47}$$

$$\mathbf{a}_{mi} \triangleq {}^{I_{k+i}}\mathbf{a} \tag{48}$$

We can also use the averaged readings for the $\boldsymbol{\omega}_i$ and $\mathbf{a}_i$, that is:

$$\boldsymbol{\omega}_{mi} = \frac{{}^{I_{k+i}}\boldsymbol{\omega} + {}^{I_{k+i+1}}\boldsymbol{\omega}}{2} \tag{49}$$

$$\mathbf{a}_{mi} = \frac{{}^{I_{k+i}}\mathbf{a} + {}^{I_{k+i+1}}\mathbf{a}}{2} \tag{50}$$

2.2 Implementation for ACI^2 MODEL 1

We summarize the steps for implementing the ACI^2 MODEL 1 as the following:

- Compute the $\hat{\boldsymbol{\omega}}_i$ and $\hat{\mathbf{a}}_i$ based on current bias estimates:

$$\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_{mi} - \hat{\mathbf{b}}_{gk} \tag{51}$$

$$\hat{\mathbf{a}}_i = \mathbf{a}_{mi} - \hat{\mathbf{b}}_{ak} \tag{52}$$

- Compute the 4 integration components Ξ_1, Ξ_2, Ξ_3 and Ξ_4:

$$\Xi_1 = \int_{t_{k+i}}^{t_{k+i+1}} {}^{I_{k+i}}_{I_\tau}\mathbf{R} d\tau \tag{53}$$

$$\Xi_2 = \int_{t_{k+i}}^{t_{k+i+1}} \int_{t_{k+i}}^{t_s} {}^{I_{k+i}}_{I_\tau}\mathbf{R} d\tau ds \tag{54}$$

$$\Xi_3 = \int_{t_{k+i}}^{t_{k+i+1}} {}^{I_{k+i}}_{I_\tau}\hat{\mathbf{R}} \lfloor \hat{\mathbf{a}}_i \rfloor \mathbf{J}_r(\hat{\boldsymbol{\omega}}_i \delta\tau)\delta\tau d\tau \tag{55}$$

$$\Xi_4 = \int_{t_{k+i}}^{t_{k+i+1}} \int_{t_{k+i}}^{t_s} {}^{I_{k+i}}_{I_\tau}\hat{\mathbf{R}} \lfloor \hat{\mathbf{a}}_i \rfloor \mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i \delta\tau\right)\delta\tau d\tau ds \tag{56}$$

  The 4 integration components can be evaluated by RK4 or by our analytic solution in Appendix A.

- Compute the integrated measurements:

$$\Delta\hat{\bar{q}}_{i+1} = \Delta\hat{\bar{q}}_i \otimes \bar{q}(\hat{\boldsymbol{\omega}}_i \delta t_i) \tag{57}$$

$$\Delta\hat{\mathbf{v}}_{i+1} = \Delta\hat{\mathbf{v}}_i + {}^{I_k}_{I_{k+i}}\hat{\mathbf{R}}\Xi_1 \cdot \hat{\mathbf{a}}_i \tag{58}$$

$$\Delta\hat{\mathbf{p}}_{i+1} = \Delta\hat{\mathbf{p}}_i + \Delta\hat{\mathbf{v}}_i \delta t_i + {}^{I_k}_{I_{k+i}}\hat{\mathbf{R}}\Xi_2 \cdot \hat{\mathbf{a}}_i \tag{59}$$

  where $\Delta\hat{\mathbf{R}}_i \triangleq {}^{I_k}_{I_{k+i}}\hat{\mathbf{R}}$, corresponds to $\Delta\hat{\bar{q}}_i$. The integrated measurements starts from zero rotation, zero position and zero velocity.

- Compute the Bias Jacobians:

$$\mathbf{H}^q_{bg_{i+1}} = {}^{I_{k+i}}_{I_{k+i+1}}\hat{\mathbf{R}}^\top \mathbf{H}^q_{bg_i} - \mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i \delta t_i\right)\delta t_i \tag{60}$$

$$\mathbf{H}^v_{bg_{i+1}} = \mathbf{H}^v_{bg_i} - \Delta\hat{\mathbf{R}}_i \lfloor \Xi_1 \hat{\mathbf{a}}_i \rfloor \mathbf{H}^q_{bg_i} + \Delta\hat{\mathbf{R}}_i \Xi_3 \tag{61}$$

$$\mathbf{H}^v_{ba_{i+1}} = \mathbf{H}^v_{ba_i} - \Delta\hat{\mathbf{R}}_i \Xi_1 \tag{62}$$

$$\mathbf{H}^p_{bg_{i+1}} = \mathbf{H}^p_{bg_i} + \mathbf{H}^v_{bg_i}\delta t_i - \Delta\hat{\mathbf{R}}_i \lfloor \Xi_2 \hat{\mathbf{a}}_i \rfloor \mathbf{H}^q_{bg_i} + \Delta\hat{\mathbf{R}}_i \Xi_4 \tag{63}$$

$$\mathbf{H}^p_{ba_{i+1}} = \mathbf{H}^p_{ba_i} + \mathbf{H}^v_{ba_i}\delta t_i - \Delta\hat{\mathbf{R}}_i \Xi_2 \tag{64}$$

  where $\mathbf{H}^y_x$ represents the measurement y to state x Jacobians and starts from $\mathbf{0}$.

- Compute the measurement state transition matrix:

$$\mathbf{\Phi}(i+1, i) = \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{14} & \mathbf{0}_3 \\ \mathbf{\Phi}_{21} & \mathbf{I}_3 & \mathbf{I}_3\delta t_i & \mathbf{\Phi}_{24} & \mathbf{\Phi}_{25} \\ \mathbf{\Phi}_{31} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{34} & \mathbf{\Phi}_{35} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{65}$$

  with:

$$\mathbf{\Phi}_{11} = {}^{I_{k+i}}_{I_{k+i+1}}\hat{\mathbf{R}}^\top, \ \mathbf{\Phi}_{14} = -\mathbf{J}_r\left(\hat{\omega}_i \delta t_i\right)\delta t_i \tag{66}$$

$$\mathbf{\Phi}_{21} = -\Delta\hat{\mathbf{R}}_i \lfloor \Xi_2 \hat{\mathbf{a}}_i \rfloor, \ \mathbf{\Phi}_{24} = \Delta\hat{\mathbf{R}}_i \Xi_4, \ \mathbf{\Phi}_{25} = -\Delta\hat{\mathbf{R}}_i \Xi_2 \tag{67}$$

$$\mathbf{\Phi}_{31} = -\Delta\hat{\mathbf{R}}_i \lfloor \Xi_1 \hat{\mathbf{a}}_i \rfloor, \ \mathbf{\Phi}_{34} = \Delta\hat{\mathbf{R}}_i \Xi_3, \ \mathbf{\Phi}_{35} = -\Delta\hat{\mathbf{R}}_i \Xi_1 \tag{68}$$

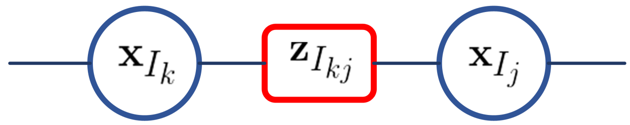

Figure 3: The analytic combine IMU (ACI) factor connecting the IMU state $\mathbf{x}_{I_k}$ and $\mathbf{x}_{I_j}$

- Compute the measurement noise Jacobian matrix:

$$\mathbf{G}_i = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t_i & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t_i \end{bmatrix} \tag{69}$$

with:

$$\mathbf{G}_{11} = -\mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i \delta t_i\right)\delta t_i \tag{70}$$

$$\mathbf{G}_{21} = \Delta\hat{\mathbf{R}}_i \Xi_4, \;\; \mathbf{G}_{22} = -\Delta\hat{\mathbf{R}}_i \Xi_2 \tag{71}$$

$$\mathbf{G}_{31} = \Delta\hat{\mathbf{R}}_i \Xi_3, \;\; \mathbf{G}_{32} = -\Delta\hat{\mathbf{R}}_i \Xi_1 \tag{72}$$

- Compute the measurement covariance:

$$\mathbf{Q}_{i+1} = \boldsymbol{\Phi}(i+1,i)\mathbf{Q}_i\boldsymbol{\Phi}^\top(i+1,i) + \mathbf{G}_i\mathbf{Q}_{di}\mathbf{G}_i^\top \tag{73}$$

with initial $\mathbf{Q}_0 = \mathbf{0}$ and:

$$\mathbf{Q}_{di} = \begin{bmatrix} \sigma_{gdi}^2\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \sigma_{adi}^2\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \sigma_{wgdi}^2\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \sigma_{wadi}^2\mathbf{I}_3 \end{bmatrix} \tag{74}$$

where $\sigma_{gdi}^2 = \sigma_g^2/\delta t_i$, $\sigma_{adi}^2 = \sigma_a^2/\delta t_i$, $\sigma_{wgdi}^2 = \sigma_{wg}^2/\delta t_i$ and $\sigma_{wadi}^2 = \sigma_{wa}^2/\delta t_i$.

2.3 IMU Cost Function Based on ACI² MODEL 1

The analytic combined IMU factor can be shown in Fig. 3.

From time interval t_k to t_{k+i+1}, the integrated measurements can be defined as:

$$\underbrace{\begin{bmatrix} \Delta\bar{q}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{n}_I) \\ \Delta\mathbf{p}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{n}_I) \\ \Delta\mathbf{v}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{n}_I) \\ \Delta\mathbf{b}_{g_{i+1}}(\mathbf{n}_I) \\ \Delta\mathbf{b}_{a_{i+1}}(\mathbf{n}_I) \end{bmatrix}}_{\mathbf{z}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{n}_I)} = \underbrace{\begin{bmatrix} \mathbf{h}_q(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}) \otimes \bar{q}^{-1}(\mathbf{H}^q_b\tilde{\mathbf{x}}_{\mathbf{b}_k}) \otimes \bar{q}^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_{\mathbf{b}_k}) \\ \mathbf{h}_p(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}) - \mathbf{H}^p_b\Delta\mathbf{x}_{\mathbf{b}_k} - \mathbf{H}^p_b\tilde{\mathbf{x}}_{\mathbf{b}_k} \\ \mathbf{h}_v(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}) - \mathbf{H}^v_b\Delta\mathbf{x}_{\mathbf{b}_k} - \mathbf{H}^v_b\tilde{\mathbf{x}}_{\mathbf{b}_k} \\ \mathbf{b}_{g_j} - \mathbf{b}_{g_k} \\ \mathbf{b}_{a_j} - \mathbf{b}_{a_k} \end{bmatrix}}_{\mathbf{h}(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}, \mathbf{x}_{\mathbf{b}_k}, \mathbf{x}_{\mathbf{b}_j})} \tag{75}$$

Since the linearization point for $\mathbf{x}_{\mathbf{b}_k}$ is fixed, so, we can build the following IMU factor as:

$$\arg\min_{\mathbf{x}_{I_k}, \mathbf{x}_{I_j}} \left\| \mathbf{z}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{0}) \boxminus \mathbf{h}(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}, \mathbf{x}_{\mathbf{b}_k}, \mathbf{x}_{\mathbf{b}_j}) \right\|^2_{\mathbf{Q}^{-1}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k})} \tag{76}$$

$$\Rightarrow \arg\min_{\mathbf{x}_{I_k}, \mathbf{x}_{I_j}} \left\| \tilde{\mathbf{z}}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, \mathbf{0}) - \mathbf{H}^0_{\mathbf{x}_{bk}}\Delta\hat{\mathbf{x}}^{(0)}_{bk} - \begin{bmatrix} \mathbf{H}_{\mathbf{x}_{nk}} & \mathbf{H}^0_{\mathbf{x}_{bk}} & \mathbf{H}_{\mathbf{x}_{nj}} & \mathbf{H}_{\mathbf{x}_{bj}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_{I_k} \\ \tilde{\mathbf{x}}_{I_j} \end{bmatrix} \right\|^2_{\mathbf{Q}^{-1}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k})} \tag{77}$$

where $\mathbf{H}^0_{\mathbf{x}_{bk}}$ and $\mathbf{Q}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k})$ are both computed at the initial bias estimate $\hat{\mathbf{x}}^0_{\mathbf{b}_k}$. Based on the integrated componetes, the predicted state $\mathbf{x}_{I_j}$ can be computed as:

$$
\begin{aligned}
{}^G_{I_j}\hat{\bar{q}} &= {}^G_{I_k}\hat{\bar{q}} \otimes \Delta\hat{\bar{q}}_{i+1} && (78)\\
{}^G\hat{\mathbf{p}}_{I_j} &= {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{v}}_{I_k}\Delta t_{i+1} + {}^G_{I_k}\hat{\mathbf{R}}\Delta\hat{\mathbf{p}}_{i+1} + \frac{1}{2}{}^G\mathbf{g}\Delta t^2_{i+1} && (79)\\
{}^G\hat{\mathbf{v}}_{I_j} &= {}^G\hat{\mathbf{v}}_{I_k} + {}^G_{I_k}\hat{\mathbf{R}}\Delta\hat{\mathbf{v}}_{i+1} + {}^G\mathbf{g}\Delta t_{i+1} && (80)\\
\hat{\mathbf{b}}_{g_{I_j}} &= \hat{\mathbf{b}}_{g_{I_k}} && (81)\\
\hat{\mathbf{b}}_{a_{I_j}} &= \hat{\mathbf{b}}_{a_{I_k}} && (82)
\end{aligned}
$$

where ${}^G\mathbf{g} = \begin{bmatrix}0 & 0 & -g\end{bmatrix}^\top$. The measurement Jacobians for state $\mathbf{x}_k$ as:

$$
\frac{\partial\tilde{\mathbf{z}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} = \begin{bmatrix} \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_k}} \end{bmatrix} = \begin{bmatrix}
\frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_k}}
\end{bmatrix} \tag{83}
$$

$$
= \begin{bmatrix}
-\mathbf{J}_r^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_b){}^G_{I_j}\hat{\mathbf{R}}^\top {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{J}_r^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_b)\mathbf{H}^q_{bg} & \mathbf{0}_3 \\
\lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_{I_j} - {}^G\hat{\mathbf{p}}_{I_k} - {}^G\hat{\mathbf{v}}_{I_k}\Delta t_{i+1} - \frac{1}{2}{}^G\mathbf{g}\Delta t^2_{i+1})\rfloor & -{}^G_{I_k}\hat{\mathbf{R}}^\top & -{}^G_{I_k}\hat{\mathbf{R}}^\top\Delta t_{i+1} & -\mathbf{H}^p_{bg} & -\mathbf{H}^p_{ba} \\
\lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top\left({}^G\hat{\mathbf{v}}_{I_j} - {}^G\hat{\mathbf{v}}_{I_k} - {}^G\mathbf{g}\Delta t_{i+1}\right)\rfloor & \mathbf{0}_3 & -{}^G_{I_k}\hat{\mathbf{R}}^\top & -\mathbf{H}^v_{bg} & -\mathbf{H}^v_{ba} \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3
\end{bmatrix} \tag{84}
$$

The measurement Jacobians for state $\mathbf{x}_j$ as:

$$
\frac{\partial\tilde{\mathbf{z}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_j}} = \begin{bmatrix} \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_j}} \end{bmatrix} = \begin{bmatrix}
\frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\delta\boldsymbol{\theta}_j} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_j}} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_j}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_j}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_j}} \\
\frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\delta\boldsymbol{\theta}_j} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_j}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_j}} \\
\frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\delta\boldsymbol{\theta}_j} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_j}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_j}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\delta\boldsymbol{\theta}_j} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_j}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\delta\boldsymbol{\theta}_j} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_j}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_j}}
\end{bmatrix} \tag{85}
$$

$$
= \begin{bmatrix}
\mathbf{J}_r^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_b) & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3
\end{bmatrix} \tag{86}
$$

3 ACI^2 MODEL 2

In this section, we present the derivations for analytic combined IMU integrator (ACI^2) MODEL 2. MODEL 2 assumes the true acceleration is constant between consecutive sampling intervals.

3.1 Implementation for ACI^2 MODEL 2

We summarize the steps for implementing the ACI^2 MODEL 2 as the following:

- Compute the $\hat{\boldsymbol{\omega}}_i$ and $\hat{\mathbf{a}}_i$ based on current bias estimates:

$$\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_{mi} - \hat{\mathbf{b}}_{gk} \tag{87}$$

$$\hat{\mathbf{a}}_i = \mathbf{a}_{mi} - \hat{\mathbf{b}}_{ak} + {}^{I_{k+i}}_{I_k}\hat{\mathbf{R}}{}^{I_k}_G\hat{\mathbf{R}}^{0G}\mathbf{g} \tag{88}$$

  where ${}^{I_k}_G\hat{\mathbf{R}}^0$ denotes the initial estimate of ${}^{I_k}_G\hat{\mathbf{R}}$ used for this integration.

- Compute the 4 integration components Ξ_1, Ξ_2, Ξ_3 and Ξ_4:

$$\Xi_1 = \int_{t_{k+i}}^{t_{k+i+1}} {}^{I_{k+i}}_{I_\tau}\mathbf{R}d\tau \tag{89}$$

$$\Xi_2 = \int_{t_{k+i}}^{t_{k+i+1}} \int_{t_{k+i}}^{t_s} {}^{I_{k+i}}_{I_\tau}\mathbf{R}d\tau ds \tag{90}$$

$$\Xi_3 = \int_{t_{k+i}}^{t_{k+i+1}} {}^{I_{k+i}}_{I_\tau}\hat{\mathbf{R}}\lfloor\hat{\mathbf{a}}_i\rfloor\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\delta\tau)\delta\tau d\tau \tag{91}$$

$$\Xi_4 = \int_{t_{k+i}}^{t_{k+i+1}} \int_{t_{k+i}}^{t_s} {}^{I_{k+i}}_{I_\tau}\hat{\mathbf{R}}\lfloor\hat{\mathbf{a}}_i\rfloor\mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i\delta\tau\right)\delta\tau d\tau ds \tag{92}$$

  The 4 integration components can be evaluated by RK4 or by our analytic solution in Appendix A.

- Compute the integrated measurements:

$$\Delta\hat{\bar{q}}_{i+1} = \Delta\hat{\bar{q}}_i \otimes \bar{q}(\hat{\boldsymbol{\omega}}_i\delta t_i) \tag{93}$$

$$\Delta\hat{\mathbf{v}}_{i+1} = \Delta\hat{\mathbf{v}}_i + \Delta\hat{\mathbf{R}}_i\Xi_1 \cdot \hat{\mathbf{a}}_i \tag{94}$$

$$\Delta\hat{\mathbf{p}}_{i+1} = \Delta\hat{\mathbf{p}}_i + \Delta\hat{\mathbf{v}}_i\delta t_i + \Delta\hat{\mathbf{R}}_i\Xi_2 \cdot \hat{\mathbf{a}}_i \tag{95}$$

- Compute the Bias Jacobians:

$$\mathbf{H}^q_{bg_{i+1}} = {}^{I_{k+i}}_{I_{k+i+1}}\hat{\mathbf{R}}^\top\mathbf{H}^q_{bg_i} - \mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i\delta t_i\right)\delta t_i \tag{96}$$

$$\mathbf{H}^v_{bg_{i+1}} = \mathbf{H}^v_{bg_i} - \Delta\hat{\mathbf{R}}_i\lfloor\Xi_1\hat{\mathbf{a}}_i\rfloor\mathbf{H}^q_{bg_i} + \Delta\hat{\mathbf{R}}_i\Xi_3 \tag{97}$$

$$\mathbf{H}^v_{\theta_{k_{i+1}}} = \mathbf{H}^v_{\theta_{k_i}} + \Delta\hat{\mathbf{R}}_i\Xi_1\Delta\hat{\mathbf{R}}_i^\top\lfloor{}^{I_k}_G\hat{\mathbf{R}}^{0G}\mathbf{g}\rfloor \tag{98}$$

$$\mathbf{H}^v_{ba_{i+1}} = \mathbf{H}^v_{ba_i} - \Delta\hat{\mathbf{R}}_i\Xi_1 \tag{99}$$

$$\mathbf{H}^p_{bg_{i+1}} = \mathbf{H}^p_{bg_i} + \mathbf{H}^v_{bg_i}\delta t_i - \Delta\hat{\mathbf{R}}_i\lfloor\Xi_2\hat{\mathbf{a}}_i\rfloor\mathbf{H}^q_{bg_i} + \Delta\hat{\mathbf{R}}_i\Xi_4 \tag{100}$$

$$\mathbf{H}^p_{ba_{i+1}} = \mathbf{H}^p_{ba_i} + \mathbf{H}^v_{ba_i}\delta t_i - \Delta\hat{\mathbf{R}}_i\Xi_2 \tag{101}$$

$$\mathbf{H}^p_{\theta_{k_{i+1}}} = \mathbf{H}^p_{\theta_{k_i}} + \mathbf{H}^v_{\theta_{k_i}}\delta t_i + \Delta\hat{\mathbf{R}}_i\Xi_2\Delta\hat{\mathbf{R}}_i^\top\lfloor{}^{I_k}_G\hat{\mathbf{R}}^{0G}\mathbf{g}\rfloor \tag{102}$$

- Compute the measurement state transition matrix:

$$\boldsymbol{\Phi}(i+1,i) = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \boldsymbol{\Phi}_{14} & \mathbf{0}_3 \\ \boldsymbol{\Phi}_{21} & \mathbf{I}_3 & \mathbf{I}_3\delta t_i & \boldsymbol{\Phi}_{24} & \boldsymbol{\Phi}_{25} \\ \boldsymbol{\Phi}_{31} & \mathbf{0}_3 & \mathbf{I}_3 & \boldsymbol{\Phi}_{34} & \boldsymbol{\Phi}_{35} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{103}$$

with:

$$\mathbf{\Phi}_{11} = {}^{I_{k+i}}_{I_{k+i+1}}\hat{\mathbf{R}}^{\top},\ \mathbf{\Phi}_{14} = -\mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i\delta t_i\right)\delta t_i \tag{104}$$

$$\mathbf{\Phi}_{21} = -\Delta\hat{\mathbf{R}}_i\lfloor\Xi_2\hat{\mathbf{a}}_i\rfloor + \Delta\hat{\mathbf{R}}_i\Xi_2\lfloor\Delta\hat{\mathbf{R}}_i^{\top}{}^{I_k}_{G}\hat{\mathbf{R}}^{0G}\mathbf{g}\rfloor,\ \mathbf{\Phi}_{24} = \Delta\hat{\mathbf{R}}_i\Xi_4,\ \mathbf{\Phi}_{25} = -\Delta\hat{\mathbf{R}}_i\Xi_2 \tag{105}$$

$$\mathbf{\Phi}_{31} = -\Delta\hat{\mathbf{R}}_i\lfloor\Xi_1\hat{\mathbf{a}}_i\rfloor + \Delta\hat{\mathbf{R}}_i\Xi_1\lfloor\Delta\hat{\mathbf{R}}_i^{\top}{}^{I_k}_{G}\hat{\mathbf{R}}^{0G}\mathbf{g}\rfloor,\ \mathbf{\Phi}_{34} = \Delta\hat{\mathbf{R}}_i\Xi_3,\ \mathbf{\Phi}_{35} = -\Delta\hat{\mathbf{R}}_i\Xi_1 \tag{106}$$

- Compute the measurement noise Jacobian matrix:

$$\mathbf{G}_i = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t_i & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t_i \end{bmatrix} \tag{107}$$

with:

$$\mathbf{G}_{11} = -\mathbf{J}_r\left(\hat{\boldsymbol{\omega}}_i\delta t_i\right)\delta t_i \tag{108}$$

$$\mathbf{G}_{21} = \Delta\hat{\mathbf{R}}_i\Xi_4,\ \mathbf{G}_{22} = -\Delta\hat{\mathbf{R}}_i\Xi_2 \tag{109}$$

$$\mathbf{G}_{31} = \Delta\hat{\mathbf{R}}_i\Xi_3,\ \mathbf{G}_{32} = -\Delta\hat{\mathbf{R}}_i\Xi_1 \tag{110}$$

- Compute the measurement covariance:

$$\mathbf{Q}_{i+1} = \mathbf{\Phi}(i+1,i)\mathbf{Q}_i\mathbf{\Phi}^{\top}(i+1,i) + \mathbf{G}_i\mathbf{Q}_{di}\mathbf{G}_i^{\top} \tag{111}$$

with:

$$\mathbf{Q}_{di} = \begin{bmatrix} \sigma_{gdi}^2\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \sigma_{adi}^2\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \sigma_{wgdi}^2\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \sigma_{wadi}^2\mathbf{I}_3 \end{bmatrix} \tag{112}$$

where $\sigma_{gdi}^2 = \sigma_g^2/\delta t_i$, $\sigma_{adi}^2 = \sigma_a^2/\delta t_i$, $\sigma_{wgdi}^2 = \sigma_{wg}^2/\delta t_i$ and $\sigma_{wadi}^2 = \sigma_{wa}^2/\delta t_i$.

3.2 IMU Cost Function Based on ACI^2 MODEL 2

From time interval t_k to t_{k+i+1}, the integrated measurements can be defined as:

$$\underbrace{\begin{bmatrix} \Delta\bar{q}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k},\mathbf{n}_I) \\ \Delta\mathbf{p}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k},{}^G_{I_k}\hat{\bar{q}}^0,\mathbf{n}_I) \\ \Delta\mathbf{v}_{i+1}(\hat{\mathbf{x}}^0_{\mathbf{b}_k},{}^G_{I_k}\hat{\bar{q}}^0,\mathbf{n}_I) \\ \Delta\mathbf{b}_{g_{i+1}}(\mathbf{n}_I) \\ \Delta\mathbf{b}_{a_{i+1}}(\mathbf{n}_I) \end{bmatrix}}_{\mathbf{z}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k},{}^G_{I_k}\hat{\bar{q}}^0,\mathbf{n}_I)} = \underbrace{\begin{bmatrix} \mathbf{h}_q(\mathbf{x}_{\mathbf{n}_k},\mathbf{x}_{\mathbf{n}_j})\otimes\bar{q}^{-1}(\mathbf{H}^q_b\tilde{\mathbf{x}}_{\mathbf{b}_k})\otimes\bar{q}^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_{\mathbf{b}_k}) \\ \mathbf{h}_p(\mathbf{x}_{\mathbf{n}_k},\mathbf{x}_{\mathbf{n}_j}) - \mathbf{H}^p_b\Delta\mathbf{x}_{\mathbf{b}_k} - \mathbf{H}^p_b\tilde{\mathbf{x}}_{\mathbf{b}_k} - \mathbf{H}^p_{\theta_k}\Delta\boldsymbol{\theta}_k - \mathbf{H}^p_{\theta_k}\mathbf{J}_r^{-1}(\Delta\boldsymbol{\theta}_k)\delta\boldsymbol{\theta}_k \\ \mathbf{h}_v(\mathbf{x}_{\mathbf{n}_k},\mathbf{x}_{\mathbf{n}_j}) - \mathbf{H}^v_b\Delta\mathbf{x}_{\mathbf{b}_k} - \mathbf{H}^v_b\tilde{\mathbf{x}}_{\mathbf{b}_k} - \mathbf{H}^v_{\theta_k}\Delta\boldsymbol{\theta}_k - \mathbf{H}^v_{\theta_k}\mathbf{J}_r^{-1}(\Delta\boldsymbol{\theta}_k)\delta\boldsymbol{\theta}_k \\ \mathbf{b}_{g_j} - \mathbf{b}_{g_k} \\ \mathbf{b}_{a_j} - \mathbf{b}_{a_k} \end{bmatrix}}_{\mathbf{h}(\mathbf{x}_{\mathbf{n}_k},\mathbf{x}_{\mathbf{n}_j},\mathbf{x}_{\mathbf{b}_k},\mathbf{x}_{\mathbf{b}_j})} \tag{113}$$

where ${}^G_{I_k}\hat{\mathbf{R}}$ is the current estimate of ${}^G_{I_k}\mathbf{R}$. $\Delta\boldsymbol{\theta}_k$ and its related Jacobians are defined from:

$${}^G_{I_k}\mathbf{R} = {}^G_{I_k}\hat{\mathbf{R}}^0\exp(\delta\boldsymbol{\theta}^0_k) = {}^G_{I_k}\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_k) \tag{114}$$

$$\Rightarrow \exp(\delta\boldsymbol{\theta}^0_k) = \left({}^G_{I_k}\hat{\mathbf{R}}^0\right)^{\top}{}^G_{I_k}\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_k) \tag{115}$$

$$\Rightarrow \exp(\delta\boldsymbol{\theta}^0_k) = \exp(\Delta\boldsymbol{\theta}_k)\exp(\delta\boldsymbol{\theta}_k) \tag{116}$$

$$\Rightarrow \delta\boldsymbol{\theta}^0_k \simeq \Delta\boldsymbol{\theta}_k + \mathbf{J}_r^{-1}(\Delta\boldsymbol{\theta}_k)\delta\boldsymbol{\theta}_k \tag{117}$$

Note that we define $\Delta\boldsymbol{\theta} \triangleq \mathbf{log}\left(\left({}^G_{I_k}\hat{\mathbf{R}}^0\right)^\top {}^G_{I_k}\hat{\mathbf{R}}\right)$. Since the linearization point for $\mathbf{x}_{\mathbf{b}_k}$ and ${}^G_{I_k}\bar{q}^0$ is fixed, so, we can build the following IMU factor as:

$$\arg\min_{\mathbf{x}_{I_k},\mathbf{x}_{I_j}} \left\|\mathbf{z}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, {}^G_{I_k}\hat{\bar{q}}^0, \mathbf{0}) \boxminus \mathbf{h}(\mathbf{x}_{\mathbf{n}_k}, \mathbf{x}_{\mathbf{n}_j}, \mathbf{x}_{\mathbf{b}_k}, \mathbf{x}_{\mathbf{b}_j})\right\|^2_{\mathbf{Q}^{-1}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, {}^G_{I_k}\hat{\bar{q}}^0)} \tag{118}$$

$$\Rightarrow \arg\min_{\mathbf{x}_{I_k},\mathbf{x}_{I_j}} \left\|\tilde{\mathbf{z}}_{I_{kj}}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, {}^G_{I_k}\hat{\bar{q}}^0, \mathbf{0}) - \begin{bmatrix}\mathbf{H}_{\mathbf{x}_{nk}} & \mathbf{H}^0_{\mathbf{x}_{bk}} & \mathbf{H}_{\mathbf{x}_{nj}} & \mathbf{H}_{\mathbf{x}_{bj}}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{x}}_{I_k} \\ \tilde{\mathbf{x}}_{I_j}\end{bmatrix}\right\|^2_{\mathbf{Q}^{-1}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, {}^G_{I_k}\hat{\bar{q}}^0)} \tag{119}$$

where $\mathbf{H}^0_{\mathbf{x}_{bk}}$ and $\mathbf{Q}_{kj}(\hat{\mathbf{x}}^0_{\mathbf{b}_k}, {}^G_{I_k}\hat{\bar{q}}^0)$ are both computed at the initial bias estimate $\hat{\mathbf{x}}^0_{\mathbf{b}_k}$ and ${}^G_{I_k}\hat{\bar{q}}^0$. Based on the integrated componets, the predicted state $\mathbf{x}_{I_j}$ can be computed as:

$$\begin{aligned}
{}^G_{I_j}\hat{\bar{q}} &= {}^G_{I_k}\hat{\bar{q}} \otimes \Delta\hat{\bar{q}}_{i+1} && (120)\\
{}^G\hat{\mathbf{p}}_{I_j} &= {}^G\hat{\mathbf{p}}_{I_k} + {}^G\hat{\mathbf{v}}_{I_k}\Delta t_{i+1} + {}^G_{I_k}\hat{\mathbf{R}}\Delta\hat{\mathbf{p}}_{i+1} && (121)\\
{}^G\hat{\mathbf{v}}_{I_j} &= {}^G\hat{\mathbf{v}}_{I_k} + {}^G_{I_k}\hat{\mathbf{R}}\Delta\hat{\mathbf{v}}_{i+1} && (122)\\
\hat{\mathbf{b}}_{g_{I_j}} &= \hat{\mathbf{b}}_{g_{I_k}} && (123)\\
\hat{\mathbf{b}}_{a_{I_j}} &= \hat{\mathbf{b}}_{a_{I_k}} && (124)
\end{aligned}$$

The measurement Jacobians for state $\mathbf{x}_k$ as:

$$\frac{\partial\tilde{\mathbf{z}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} = \begin{bmatrix} \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_k}} \\ \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{x}}_{I_k}} \end{bmatrix} = \begin{bmatrix}
\frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{p}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{v}}_{i+1}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_k}} \\
\frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{p}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{v}}_{I_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{g_k}} & \frac{\partial\Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial\tilde{\mathbf{b}}_{a_k}}
\end{bmatrix} \tag{125}$$

$$= \begin{bmatrix}
-\mathbf{J}_r^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_b){}^G_{I_j}\hat{\mathbf{R}}^\top{}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{J}_r^{-1}(\mathbf{H}^q_b\Delta\mathbf{x}_b)\mathbf{H}^q_{bg} & \mathbf{0}_3 \\
\lfloor{}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_{I_j} - {}^G\hat{\mathbf{p}}_{I_k} - {}^G\hat{\mathbf{v}}_{I_k}\Delta t_{i+1})\rfloor - \mathbf{H}^p_{\theta_k}\mathbf{J}_r^{-1}(\Delta\boldsymbol{\theta}_k) & -{}^G_{I_k}\hat{\mathbf{R}}^\top & -{}^G_{I_k}\hat{\mathbf{R}}^\top\Delta t_{i+1} & -\mathbf{H}^p_{bg} & -\mathbf{H}^p_{ba} \\
\lfloor{}^G_{I_k}\hat{\mathbf{R}}^\top\left({}^G\hat{\mathbf{v}}_{I_j} - {}^G\hat{\mathbf{v}}_{I_k}\right)\rfloor - \mathbf{H}^v_{\theta_k}\mathbf{J}_r^{-1}(\Delta\boldsymbol{\theta}_k) & \mathbf{0}_3 & -{}^G_{I_k}\hat{\mathbf{R}}^\top & -\mathbf{H}^v_{bg} & -\mathbf{H}^v_{ba} \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3
\end{bmatrix} \tag{126}$$

The measurement Jacobians for state $\mathbf{x}_j$ as:

$$\frac{\partial \tilde{\mathbf{z}}_{i+1}}{\partial \tilde{\mathbf{x}}_{I_j}} = \begin{bmatrix} \frac{\partial \delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \tilde{\mathbf{x}}_{I_j}} \\ \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \tilde{\mathbf{x}}_{I_j}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \delta\boldsymbol{\theta}_j} & \frac{\partial \delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \tilde{\mathbf{p}}_{I_j}} & \frac{\partial \delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \tilde{\mathbf{v}}_{I_j}} & \frac{\partial \Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \tilde{\mathbf{b}}_{g_j}} & \frac{\partial \Delta\Delta\boldsymbol{\theta}_{i+1}}{\partial \tilde{\mathbf{b}}_{a_j}} \\ \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \delta\boldsymbol{\theta}_j} & \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \tilde{\mathbf{p}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \tilde{\mathbf{v}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \tilde{\mathbf{b}}_{g_j}} & \frac{\partial \Delta\tilde{\mathbf{p}}_{i+1}}{\partial \tilde{\mathbf{b}}_{a_j}} \\ \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \delta\boldsymbol{\theta}_j} & \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \tilde{\mathbf{p}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \tilde{\mathbf{v}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \tilde{\mathbf{b}}_{g_j}} & \frac{\partial \Delta\tilde{\mathbf{v}}_{i+1}}{\partial \tilde{\mathbf{b}}_{a_j}} \\ \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \delta\boldsymbol{\theta}_j} & \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \tilde{\mathbf{p}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \tilde{\mathbf{v}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \tilde{\mathbf{b}}_{g_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{g_{i+1}}}{\partial \tilde{\mathbf{b}}_{a_j}} \\ \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \delta\boldsymbol{\theta}_j} & \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \tilde{\mathbf{p}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \tilde{\mathbf{v}}_{I_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \tilde{\mathbf{b}}_{g_j}} & \frac{\partial \Delta\tilde{\mathbf{b}}_{a_{i+1}}}{\partial \tilde{\mathbf{b}}_{a_j}} \end{bmatrix} \tag{127}$$

$$= \begin{bmatrix} \mathbf{J}_r^{-1}(\mathbf{H}_b^q \Delta\mathbf{x}_b) & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{128}$$

4 Visual Measurement Jacobians

The measurement Jacobians for the visual model can be written as:

$$\frac{\partial {}^C\tilde{\mathbf{p}}_f}{\partial \delta\boldsymbol{\theta}_k} = {}^C_I\hat{\mathbf{R}} \lfloor {}^{I_k}_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}\right) \rfloor \tag{129}$$

$$\frac{\partial {}^C\tilde{\mathbf{p}}_f}{\partial {}^G\tilde{\mathbf{p}}_{I_k}} = -{}^C_I\hat{\mathbf{R}} {}^{I_k}_G\hat{\mathbf{R}} \tag{130}$$

$$\frac{\partial {}^C\tilde{\mathbf{p}}_f}{\partial \tilde{t}_d} = {}^C_I\hat{\mathbf{R}} \left(\lfloor {}^{I_k}\hat{\boldsymbol{\omega}} \rfloor {}^{I_k}_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}\right) + {}^{I_k}_G\hat{\mathbf{R}} {}^G\hat{\mathbf{v}}_{I_k} \right) \tag{131}$$

where ${}^G\mathbf{p}_f$ denotes the 3D feature position in global frame.

Appendix A: Integration Components for ACI^2

The first integration we need is:

$$\boldsymbol{\Xi}_1 = \mathbf{I}_3 \delta t_i + \frac{1 - \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i} \lfloor \hat{\mathbf{k}}_i \rfloor + \left(\delta t_i - \frac{\sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i} \right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{132}$$

The second integration we need is:

$$\boldsymbol{\Xi}_2 = \frac{1}{2}\delta t_i^2 \mathbf{I}_3 + \frac{\hat{\omega}_i \delta t_i - \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{k}}_i \rfloor + \left(\frac{1}{2}\delta t_i^2 - \frac{1 - \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{133}$$

The third integration can be written as:

$$
\begin{aligned}
\boldsymbol{\Xi}_3 = & \frac{1}{2}\delta t_i^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{1}{2}\delta t_i^2 - \frac{1 - \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 \\
& + \left(\frac{1}{2}\delta t_i^2 + \frac{1 - \cos(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{1}{2}\delta t_i^2 + \frac{1 - \cos(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \\
& - \frac{3\sin(\hat{\omega}_i \delta t_i) - 2\hat{\omega}_i \delta t_i - \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2
\end{aligned} \tag{134}
$$

The fourth integration we need is:

$$
\begin{aligned}
\boldsymbol{\Xi}_4 = & \frac{1}{6}\delta t_i^3 \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{2(1 - \cos(\hat{\omega}_i \delta t_i)) - (\hat{\omega}_i^2 \delta t_i^2)}{2\hat{\omega}_i^3} \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \left(\frac{2(1 - \cos(\hat{\omega}_i \delta t_i)) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \right) \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i}{\hat{\omega}_i^3} + \frac{\delta t_i^3}{6} \right) \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 \\
& + \frac{\hat{\omega}_i \delta t_i - 2\sin(\hat{\omega}_i \delta t_i) + \frac{1}{6}(\hat{\omega}_i \delta t_i)^3 + \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \frac{\hat{\omega}_i \delta t_i - 2\sin(\hat{\omega}_i \delta t_i) + \frac{1}{6}(\hat{\omega}_i \delta t_i)^3 + \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \frac{4\cos(\hat{\omega}_i \delta t_i) - 4 + (\hat{\omega}_i \delta t_i)^2 + \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2
\end{aligned} \tag{135}
$$

When $\hat{\omega}_i$ is too small, in order to avoid numerical instability, we can compute the above integration identities as:

$$\lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_1 = \delta t_i \mathbf{I}_3 + \delta t_i \sin(\hat{\omega}_i \delta t_i) \lfloor \hat{\mathbf{k}}_i \rfloor + \delta t_i \left(1 - \cos(\hat{\omega}_i \delta t_i)\right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{136}$$

$$\lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_2 = \frac{\delta t_i^2}{2} \mathbf{I}_3 + \frac{\delta t_i^2}{2} \sin(\hat{\omega}_i \delta t_i) \lfloor \hat{\mathbf{k}}_i \rfloor + \frac{\delta t_i^2}{2} \left(1 - \cos(\hat{\omega}_i \delta t_i)\right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{137}$$

$$= \frac{\delta t_i}{2} \lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_1 \tag{138}$$

$$\begin{aligned} \lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_3 = & \frac{\delta t_i^2}{2} \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\delta t_i^2 \sin(\hat{\omega}_i \delta t_i)}{2} \left(-\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor + \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2 \right) \\ & + \frac{\delta t_i^2}{2} (1 - \cos(\hat{\omega}_i \delta t_i)) \left(\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 + \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \right) \end{aligned} \tag{139}$$

$$\begin{aligned} \lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_4 = & \frac{\delta t_i^3}{6} \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\delta t_i^3 \sin(\hat{\omega}_i \delta t_i)}{6} \left(-\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor + \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2 \right) \\ & + \frac{\delta t_i^3}{6} (1 - \cos(\hat{\omega}_i \delta t_i)) \left(\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 + \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \right) \end{aligned} \tag{140}$$

$$= \frac{\delta t_i}{3} \lim_{\hat{\omega}_i \to 0} \boldsymbol{\Xi}_3 \tag{141}$$

References

[1] Joan Sola. "Quaternion kinematics for the error-state Kalman filter". In: *arXiv preprint arXiv:1711.02508* (2017).

[2] Nikolas Trawny and Stergios I. Roumeliotis. *Indirect Kalman Filter for 3D Attitude Estimation.* Tech. rep. University of Minnesota, Dept. of Comp. Sci. & Eng., Mar. 2005.

[3] Frank Chongwoo Park. "The Optimal Kinematic Design of Mechanisms". UMI Order No. GAX91-32016. PhD thesis. Cambridge, MA, USA, 1991.

[4] Gregory S Chirikjian. *Stochastic Models, Information Theory, and Lie Groups, Volume 2: Analytic Methods and Modern Applications.* Vol. 2. Springer Science & Business Media, 2011.